21-256 Exam 2: Partial Differentiation (sample)

Wednesday 18th June, 10:30–11:20am

Instructions (please read carefully)

This test is split into three sections of four questions each:

- **Section A:** Partial differentiation and the chain rule
- **Section B:** Tangent planes and linear approximation
- **Section C:** Unconstrained extrema

You should attempt **exactly ten** questions, including **at least three** questions from each section. All questions will be marked out of 10. The duration of the test is 50 minutes. Please write legibly in the blue book provided. Calculators and other electronic devices are not permitted.

Please indicate on the table below which questions you attempted.

Question	**Attempted? (✓)**	**Score**
A1		
A2		
A3		
A4		
B1		
B2		
B3		
B4		
C1		
C2		
C3		
C4		
Total	—	

Name: ______________________________

Section A

A1. Find the partial derivatives of

$$h(x, y, z) = \sqrt{x^2y^2z^2 + (x^2y + y^2z + z^2x)^2} + x^3y^3z^3 \sin(x^2y + y^2z + z^2x)$$

with respect to x, y and z.

A2. Find a function g of variables x, y such that $\dfrac{\partial g}{\partial x} = -1$, $\dfrac{\partial g}{\partial y} = 3$ and $g(1, 1) = 4$.

A3. A bug is crawling on the xy-plane in such a way that, after t seconds, its position is $(\sqrt{1+t}, 2 + \frac{1}{3}t)$. The temperature at the point (x, y) is $T(x, y)$. If $\dfrac{\partial T}{\partial x}(2, 3) = 4$ and $\dfrac{\partial T}{\partial y}(2, 3) = 3$, how fast is the temperature increasing on the bug's path after 3 seconds?

A4. Let $z = f(x, y)$, where $x = s + t$ and $y = s - t$. Show that $\left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2 = \dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t}$.

Section B

B1. Given that $\ln(\pi) \approx 1.14$, use linear approximation to estimate the value of $\sin(0.24e^{1.14}) + \cos(0.24e^{1.14})$.

B2. Show that the functions $f(x, y) = 2e^{2x}\cos(3y)$ and $g(x, y) = (1 + x)^2(2 - y^2)$ have the same linearization at the point $(0, 0)$.

B3. Find the direction of the line of intersection of the tangent planes to the surfaces

$$z = \frac{x\cos(y) + y\cos(x)}{\sqrt{3}} \qquad \text{and} \qquad 2\sin(x) + 2\sin(y) + 2\sin(z) = 3$$

at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$.

B4. Find the angle between the tangent planes of the surfaces

$$x^2y + y^2z + z^2x = 1 \qquad \text{and} \qquad xyz + xy + yz + zx = -1$$

at the point $(1, -1, 1)$.

Section C on next page

Section C

C1. Find and classify the local extrema of the function $h(x,y) = xy + x^{-1} + y^{-1}$.

C2. Find values of a such that the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & a \\ 0 & a & 8 \end{pmatrix}$ is positive-definite. Deduce that the function $g(x,y,z) = x^2 + y^2 + 4z^2 - xy - yz + 11z - 2$ has a local minimum at $(-\frac{1}{2}, -1, -\frac{3}{2})$.

C3. Find the dimensions of the (closed) box of minimal surface area that has volume $1000\,\mathrm{m}^3$.

C4. Find the global extrema of the function $f(x,y) = x^3 + y^3$ on the compact set $x^2 + y^2 \leqslant 1$.

Please submit this question sheet with your answer booklet.