21-256: Dot and cross products

Clive Newstead, Thursday 22nd May 2014

This is a summary of the important results about dot and cross products that you should know.

Dot product

The dot product $\mathbf{v} \cdot \mathbf{w}$ of two n-dimensional vectors $\mathbf{v}$ and $\mathbf{w}$ is a scalar, defined by

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Some useful facts about dot products include:

(a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$;

(b) $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w})$;

(c) $\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \cdot \mathbf{w_1} + \mathbf{v} \cdot \mathbf{w}_2$;

(d) $\mathbf{0} \cdot \mathbf{v} = 0$;

(e) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

The dot product can be interpreted geometrically: if θ is the angle between vectors $\mathbf{v}$ and $\mathbf{w}$ then

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$$

Since the inverse cosine function returns only acute angles, this means that given two vectors $\mathbf{v}$ and $\mathbf{w}$, the acute angle between them is given by

$$\cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}\right)$$

Thus $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{w} = 0$.

Projections of vectors

We can use the dot product to find the vector and scalar projections of vectors. Indeed:

- The vector projection of $\mathbf{v}$ onto $\mathbf{w} \neq \mathbf{0}$ is

$$\mathrm{proj}_{\mathbf{w}}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2}\right) \mathbf{w}$$

- The scalar projection of $\mathbf{v}$ onto $\mathbf{w} \neq \mathbf{0}$ is

$$\mathrm{comp}_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|} = \|\mathrm{proj}_{\mathbf{w}}(\mathbf{v})\|$$

Cross product

The cross product $\mathbf{v} \times \mathbf{w}$ of two 3-dimensional vectors $\mathbf{v}$ and $\mathbf{w}$ is a vector, defined by

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - w_2 v_3 \\ v_3 w_1 - w_3 v_1 \\ v_1 w_2 - w_1 v_2 \end{pmatrix}$$

Some useful facts about cross products include:

(a) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$, **not** $\mathbf{w} \times \mathbf{v}$; in particular, $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(b) $(\lambda \mathbf{v}) \times \mathbf{w} = \lambda(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (\lambda \mathbf{w})$;

(c) $\mathbf{v} \times (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v} \times \mathbf{w}_\mathbf{1} + \mathbf{v} \times \mathbf{w}_2$;

(d) $\mathbf{0} \times \mathbf{v} = \mathbf{0}$;

(e) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$; this quantity is called the *scalar triple product* and is also written $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$;

(f) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$; this quantity is called the *vector triple product*;

(g) $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

The cross product can be interpreted geometrically: if θ is the angle between vectors $\mathbf{v}$ and $\mathbf{w}$ then

$$\mathbf{v} \times \mathbf{w} = (\|\mathbf{v}\| \|\mathbf{w}\| \sin \theta)\mathbf{n}$$

where $\mathbf{n}$ is the unit vector which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$ according to the 'right-hand rule'. That is, if you point along $\mathbf{v}$ with the index finger of your right hand and $\mathbf{w}$ with the middle finger of your right hand, then $\mathbf{n}$ points in the direction of your (extended) thumb.

Parallelograms and parallelepipeds

If $\mathbf{v}$ and $\mathbf{w}$ are adjacent edges of a parallelogram, then the area A of the parallelogram is given by

$$A = \|\mathbf{v} \times \mathbf{w}\|$$

Note that $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}$ and $\mathbf{w}$ point in the same (or opposite) direction: this corresponds with the parallelogram being flat and thus having no area.

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are adjacent edges of a parallelepiped, then the volume V of the parallelepiped is given by

$$V = |[\mathbf{u}, \mathbf{v}, \mathbf{w}]| = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

Note that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = 0$ if and only if $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are coplanar: this corresponds with the parallelepiped being flat and thus having no volume.