21-128 and 15-151 problem sheet 9

Solutions to the following exercises and optional bonus problem are to be submitted through blackboard by 8:30AM on

Wednesday 30th November 2016.

Problem 1

For each example below, determine whether the given relation R is an equivalence relation on the given set S:

(a) $S = \mathbb{N} \setminus \{0, 1\}$; $(x, y) \in R$ if and only if $\gcd(x, y) > 1$.

(b) $S = \mathbb{R}$; $(x, y) \in R$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$.

Problem 2

Find the flaw in the following argument that the symmetric and transitive properties imply the reflexive property for a relation R on a set S:

> Consider $x \in S$. If $(x, y) \in R$ then the symmetric property implies that $(y, x) \in R$. Now the transitive property applied to (x, y) and (y, x) implies that $(x, x) \in R$.

Problem 3

Let x, y, z be nonnegative real numbers such that $y + z \geq 2$. Prove that

$$(x + y + z)^2 \geq 4x + 4yz$$

Problem 4

Let $P(n)$ be a mathematical statement depending on an integer n. Suppose that:

(i) $P(1)$ is true; and

(ii) Given $n \in \mathbb{N}^+$, if $P(n)$ is true, then $P(2n)$ is true; and

(iii) Given $n \in \mathbb{N}^+$ with $n > 1$, if $P(n)$ is true, then $P(n - 1)$ is true.

Prove that $P(n)$ is true for all $n \in \mathbb{N}^+$.

Problem 5

Use the result of Problem 4 to deduce that, given $n \in \mathbb{N}^+$ and nonnegative real numbers $a_1, a_2, \ldots, a_n$, that

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}$$

Hint: In step (iii) from Problem 4, let $a_n = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} a_i$ and apply $P(n)$.

Bonus Problem - (2 points)

Show that for all positive real numbers A, B, C,

$$\frac{A}{B+C} + \frac{B}{A+C} + \frac{C}{A+B} \geq \frac{3}{2}.$$