21-128 and 15-151 problem sheet 7

Solutions to the following seven exercises and optional bonus problem are to be submitted through blackboard by 8:30AM on

Thursday 10th November 2016.

Problem 1

By counting in two ways, prove that $n^2 = 2\binom{n}{2} + n$ for all $n \geq 0$.

Problem 2

By counting in two ways, prove that $\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$ for all $k, n \geq 0$.

Problem 3

By counting in two ways, prove that $\sum_{i=1}^{n} (i-1)(n-i) = \binom{n}{3}$ for all $n \geq 1$.

Problem 4

Prove that the set of all natural numbers, the set of all even natural numbers, and the set of all odd natural numbers all have the same cardinality.

Problem 5

Construct an explicit bijection from the open interval $(0, 1)$ to the interval $(0, 1]$.

Problem 6

A function $f : \mathbb{Z} \to \mathbb{Z}$ is *periodic* if there exists a positive integer k such that $f(x + k) = f(x)$ for all $x \in \mathbb{Z}$. Prove that the set of all periodic functions $\mathbb{Z} \to \mathbb{Z}$ is countable.

Problem 7

(a) A subset A of $\mathbb{R}$ has the property that, given $\varepsilon > 0$ and $x \in \mathbb{R}$, there exist $a, b \in \mathbb{R}$ with $a \in A$ and $b \notin A$, such that $|x - a| < \varepsilon$ and $|x - b| < \varepsilon$. Can A be countable? Can A be uncountable?

(b) A subset B of $\mathbb{R}$ has the property that, for every $b \in B$, there exists $\varepsilon > 0$ such that for every $x \in \mathbb{R}$, $0 < |b - x| < \varepsilon$ implies $x \notin B$. Is B countable?

[*Source: Cambridge Mathematical Tripos Part IA 2010 Exam Paper 4 Question 8.*]

Bonus Problem - (2 points)

By counting in two ways, prove that $\binom{\binom{n}{2}}{2} = 3(\binom{n}{4} + \binom{n}{3})$ for all $n \geq 3$.