

21-128 and 15-151 problem sheet 5

Solutions to the following seven exercises and optional bonus problem are to be submitted through blackboard by 8:30AM on

Thursday 13th October 2016.

There are also some practice problems, not to be turned in, for those seeking more practice and also for review prior to the exam.

Problem 1

Let p be a prime number greater than 2. Prove that $\frac{2}{p}$ can be expressed in exactly one way in the form $\frac{1}{m} + \frac{1}{n}$ where m and n are positive integers with $n > m$.

Problem 2

For $x, y, z \in \mathbb{Z}$, suppose that 5 divides $x^2 + y^2 + z^2$. Prove that 5 divides at least one of x, y or z .

Problem 3

Let m and n be positive, relatively prime integers, and r and s be integers such that $mr \equiv 1 \pmod{n}$ and $ns \equiv 1 \pmod{m}$. For fixed integers a, b , find an integer value of x in terms of a, b, m, n, r, s satisfying $x \equiv a \pmod{n}$ and $x \equiv b \pmod{m}$.

Problem 4

Let m and n be positive, relatively prime integers, and x and y be integers such that $x \equiv a \pmod{n}$ and $x \equiv b \pmod{m}$, and also $y \equiv a \pmod{n}$ and $y \equiv b \pmod{m}$ (here a and b are fixed, unknown integers). Show that $x \equiv y \pmod{mn}$.

Problem 5

The base 10 representation of an integer is *palindromic* if the digits read the same when written forward or backward. Prove that every palindromic integer with an even number of digits is divisible by 11.

Problem 6

Show your work in the following computations.

(i) Determine the unknown digits a and b in $3a27b$ given that $3a27b \equiv 1 \pmod{9}$ and $3a27b \equiv 9 \pmod{11}$.

(ii) Let p be a positive prime number. Determine the remainder left when $1^{p-1} \times 2^{p-1} \times \cdots \times (p-1)^{p-1}$ is divided by p .

(iii) Use congruence mod 9 to find the missing digit ? in $92854 \cdot 16284 = 151?034536$.

(iv) Find the remainder when 3^{100} is divided by 7.

(v) Find the remainder when 3^{100} is divided by 15.

Problem 7

Let p be a prime larger than 2. Prove that $2(p-3)! \equiv -1 \pmod{p}$. [*Hint: use Wilson's theorem.*]

Bonus Problem - 2 points

Let p be a positive prime which leaves a remainder of 1 upon division by 4. Show that p divides $((\frac{p-1}{2})!)^2 + 1$.

Extra Problem 1

1500 soldiers arrive in training camp. A few soldiers desert the camp. The drill sergeants divide the remaining soldiers into groups of five and discover there is one left over. When they divide them into groups of seven, there are three left over. When they divide them into groups of eleven, there are again three left over. Determine the number of deserters.

Extra Problem 2

Let k be an odd integer. Show that $k^2 - 1$ is divisible by 8.

Extra Problem 3

Let p be a positive prime and k be an integer. Prove that $k^2 \equiv 1 \pmod{p}$ implies ($k \equiv 1 \pmod{p}$ or $k \equiv -1 \pmod{p}$).