21-128 and 15-151 problem sheet 1

Solutions to the following seven exercises and optional bonus problem are to be submitted through blackboard by 8:30AM on

Thursday 8th September 2016.

There are also some practice problems, not to be turned in, for those seeking more practice and also for review prior to the exam.

Problem 1

Let $a, b, c \in \mathbb{Z}$. Prove that if a divides b and b divides c, then a divides c.

Problem 2

Let (x, y, z) be a Pythagorean triple, and let $P = x + y + z$ and $A = \frac{1}{2}xy$ be the perimeter and area, respectively, of the right-angled triangle whose side lengths are x, y and z.

(a) Find the possible values of (x, y, z) when $P = A$.

(b) Find the possible values of (x, y, z) when $P = 2A$.

Problem 3

For each statement below, decide whether it is true or false. Prove your claim using only properties of the natural numbers.

(a) If $n \in \mathbb{N}$ and $n^2 + (n+1)^2 = (n+2)^2$, then $n = 3$.

(b) For all $n \in \mathbb{N}$, it is false that $(n-1)^3 + n^3 = (n+1)^3$.

Problem 4

Show that $p \Leftrightarrow q$ is logically equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

Problem 5

Let $p(x, y)$ be the statement '$x + y$ is even', where x and y range over the integers.

(a) Prove that $\forall x, \exists y, p(x, y)$ is true.

(b) Prove that $\exists y, \forall x, p(x, y)$ is false.

Problem 6

(a) Show that the following statement is false:

$$\text{For all } a, x \in \mathbb{R} \text{ there is a unique } y \in \mathbb{R} \text{ such that } x^4 y + ay + x = 0$$

(b) Find the set of real numbers a such that the following statement is true:

$$\text{For all } x \in \mathbb{R} \text{ there is a unique } y \in \mathbb{R} \text{ such that } x^4 y + ay + x = 0$$

Problem 7

Let r be a rational number and let a and b be irrational numbers. Which of the following numbers is necessarily irrational?

$$a + r \quad a + b \quad ar \quad ab \quad a^r \quad r^a \quad a^b$$

Prove your claims, either by proving that the number is irrational or by providing a counterexample. If you claim that a number is irrational, then you should prove it.

Bonus Problem (2 points)

Three brilliant, flawless logicians - A, B, and C were blindfolded and each had a hat with a positive integer (possibly different for each) written on it placed on their heads.

Their blindfolds were then removed; they faced each other in a circle and each could see the hats the others were wearing, but not their own hat.

They were told that two of the numbers added up to the third. In order to be generously rewarded they needed to figure out what number was written on their hats.

Here is the conversation that took place:

Logician A: I don't know what my number is.
Logician B: I don't know what my number is.
Logician C: I don't know what my number is.
Logician A: Now I know what my number is. It is 50.

(a) What are the other numbers?

(b) What combination(s) of numbers would allow C to solve the problem in round 1?

Extra Problem 1

Let $P(x)$ be the assertion "x is odd" and let $Q(x)$ be the assertion "x is two times an integer". Determine, with proof, whether the following statements are true or false:

(a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$.

(b) $(\forall x \in \mathbb{Z})(P(x)) \Rightarrow (\forall x \in \mathbb{Z})(Q(x))$.

Extra Problem 2

In the morning section of a calculus course, 2 of the 9 women and 2 of the 10 men receive the grade of A. In the afternoon section, 6 of the 9 women and 9 of the 14 men receive an A. Verify that, in each section, a higher proportion of women than of men receive an A, but that, in the combined course, a lower proportion of women than men receive an A. Explain!

Extra Problem 3

We have two identical glasses. Glass 1 contains x ounces of wine; glass 2 contains x ounces of water ($x \geq 1$). We remove 1 ounce of wine from glass 1 and add it to glass 2. The wine and water in glass 2 mix uniformly. We now remove 1 ounce of liquid from glass 2 and add it to glass 1. Prove that the amount of water in glass 1 is now the same as the amount of wine in glass 2.