21-128 problem sheet 5

Solutions to starred (*) exercises are due at the beginning of recitation on

Thursday 15th October 2015

Please submit answers to separate questions on separate sheets of paper.

Problem 1 — 4.22 *

Verify that the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(x) = \frac{2x-1}{2x(1-x)} \quad \text{for all } x \in (0,1)$$

is a bijection.

Problem 2 — 4.29

Consider three functions $f, g, h : \mathbb{R} \to \mathbb{R}$, defined for all $x \in \mathbb{R}$ by

$$f(x) = \frac{x}{1+x^2}, \quad g(x) = \frac{x^2}{1+x^2}, \quad h(x) = \frac{x^3}{1+x^2}$$

(a) Determine which of these functions are injective.

(b) Prove that f and g are not surjective.

(c) Graph all three functions.

Problem 3 — 4.30 *

Given real numbers a, b, c, d, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (ax+by, cx+dy)$ for all $(x,y) \in \mathbb{R}^2$. Prove that f is injective if and only if f is surjective.

Problem 4 — 4.34 *

Let $f : A \to B$ and $g : B \to C$ be functions, and define $h = g \circ f$. Determine which of the following statements are true, giving proofs for the true statements and counterexamples for the false statements:

(a) If h is injective, then f is injective.

(b) If h is injective, then g is injective.

(c) If h is surjective, then f is surjective.

(d) If h is surjective, then g is surjective.

Problem 5 — 4.36

Consider functions $f : A \to B$ and $g : B \to A$. Prove that

(a) If $f \circ g$ is the identity function on B, then f is surjective.

(b) If $g \circ f$ is the identity function on A, then f is injective.

To remind you: given a set X, the identity function on X is the function $\mathrm{id}_X : X \to X$ defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Problem 6 — 4.37

Consider a function $f : A \to A$. Prove that if $f \circ f$ is injective, then f is injective.

Problem 7 — 4.45 *

Let A be a set and let $f : A \to A$ be a function. Prove that if A is finite, then f is injective if and only if f is surjective; and that if A is infinite, then this equivalence need not hold.

Problem 8 — 4.47

Prove that the set of all natural numbers, the set of all even natural numbers, and the set of all odd natural numbers all have the same cardinality.

Problem 9 — 4.51 *

Construct an explicit bijection from the open interval $(0, 1)$ to the closed interval $[0, 1]$.

Problem 10 *

Fix a prime number $p \in \mathbb{N}$, and define

$$S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$$

Prove that the function $f : S \to S$ defined for all $(x, y, z) \in S$ by

$$f(x, y, z) = \begin{cases} (x + 2z, z, y - x - z) & \text{if } x < y - z \\ (2y - x, y, x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y, x - y + z, y) & \text{if } x > 2y \end{cases}$$

is a bijection and is its own inverse (i.e. $f^{-1} = f$).

[*Optional:* Prove that if $p = 4k + 1$ for some $k \in \mathbb{N}$, then $f(x, y, z) = (x, y, z)$ for exactly one triple $(x, y, z) \in S$.]

Problem 11 *

Let $f : A \to B$ be a function.

(a) Prove that there exists a set X and functions $p : A \to X$ and $i : X \to B$, with p surjective and i injective, such that $f = i \circ p$.

(b) Prove that there exists a set Y and functions $j : A \to Y$ and $q : Y \to B$, with j injective and q surjective, such that $f = q \circ j$.