21-128 problem sheet 3

Solutions to starred (*) exercises are due at the beginning of recitation on

Thursday 24th September 2015

Please submit answers to separate questions on separate sheets of paper.

Problem 1 — 3.4 *

Let $P(n)$ be a mathematical statement depending on an integer n. Suppose that:

(i) $P(0)$ is true; and

(ii) Given $n \in \mathbb{Z}$, if $P(n)$ is true, then at least one of $P(n+1)$ or $P(n-1)$ is true.

For which $n \in \mathbb{Z}$ must $P(n)$ be true?

Problem 2 — 3.7

Find all natural numbers n such that $2n - 8 < n^2 - 8n + 17$. Prove your claim.

Problem 3 — 3.14

For each sum below, write it in summation notation and find and prove a formula in terms of n:

(a) $3 + 7 + 11 + \cdots + (4n - 1)$;

(b) $1 + 5 + 9 + \cdots + (4n + 1)$;

(c) $-1 + 2 - 3 + 4 - \cdots - (2n - 1) + 2n$;

(d) $1 - 3 + 5 - 7 + \cdots + (4n - 3) - (4n - 1)$.

Problem 4 — 3.17 *

Using induction—and without using the formulae for $\sum i$ and $\sum i^2$—prove that

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3} \text{ for all } n \in \mathbb{N}$$

Problem 5 — 3.23

Let $a \in \mathbb{R}$ with $a \neq 0$. Find the flaw in the following 'proof' that $a^n = 1$ for all $n \geq 0$:

We prove that $a^n = 1$ for all $n \geq 0$ by induction on n.

- **Basis step.** $a^0 = 1$ since any (nonzero) real number raised to the power 0 is equal to 1.
- **Induction step.** Let $n \in \mathbb{N}$ and assume the induction hypothesis. Then
$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$$

By induction, $a^n = 1$ for all $n \geq 0$. $\square$

Problem 6 — 3.27 *

Prove that
$$\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}$$
for all $n \in \mathbb{N}$.

Problem 7 — 3.33 *

Given $n \in \mathbb{N}$, let A_n be the set of intervals $[a,b] \subseteq \mathbb{R}$, where $a, b \in \mathbb{N}$ and $1 \leq a \leq b \leq n$. Find a simple formula for the number of elements of A_n for each $n \in \mathbb{N}$.

Note: this is a re-worded but equivalent form of Problem 3.33 on Page 73 of the textbook.

Problem 8 — 3.44 *

Determine the set of all natural numbers that can be expressed as the sum of some nonnegative number of 3s and some nonnegative number of 10s.

Problem 9 — 3.56 *

Let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a sequence satisfying
$$a_n = 2a_{n-1} + 3a_{n-2} \text{ for all } n \geq 3$$

(a) Given that a_1, a_2 are odd, prove that a_n is odd for all $n \in \mathbb{N}$.

(b) Given that $a_1 = a_2 = 1$, prove that $a_n = \frac{1}{2}\left(3^{n-1} - (-1)^n\right)$ for all $n \in \mathbb{N}$.

Problem 10 — 3.62

Two players alternately name dates. On each move, a player can increase the month or the day of the month, but not both. The starting position is January 1, and the player who names December 31 wins. According to the rules, the first player can start by naming some day in January after the first, or the first of some month after January.

An example play is as follows:

Jan 5, Mar 5, Mar 15, Apr 15, Apr 25, Nov 25, Nov 30, Dec 30, Dec 31

Derive a winning strategy for the first player.

Problem 11 *

Let $P(n)$ be a mathematical statement depending on an integer n. Suppose that:

(i) $P(1)$ is true; and

(ii) Given $n \in \mathbb{N}$, if $P(n)$ is true, then $P(2n)$ is true; and

(iii) Given $n \in \mathbb{N}$ with $n > 1$, if $P(n)$ is true, then $P(n-1)$ is true.

Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Deduce, given $n \in \mathbb{N}$ and nonnegative real numbers $a_1, a_2, \ldots, a_n$, that

$$\frac{1}{n}\sum_{i=1}^{n} a_i \geq \sqrt[n]{\prod_{i=1}^{n} a_i}$$

Hint: In step (iii), let $a_n = \frac{1}{n-1}\sum_{i=1}^{n-1} a_i$ and apply $P(n)$.