21-128 problem sheet 2

Solutions to starred (*) exercises are due at the beginning of recitation on

Thursday 17th September 2015

Please submit answers to separate questions on separate sheets of paper.

Problem 1 — 1.50 *

Let $f : A \to B$ be a function and let $C, D \subseteq A$.

(a) Prove that $f(C \cap D) \subseteq f(C) \cap f(D)$.

(b) Give an example to demonstrate that equality need not hold in (a).

Problem 2 — 2.4

Let $A, B \subseteq \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ and let P denote the set of positive real numbers. Without using negation words (e.g. 'not'), negate the following expressions:

(a) For all $x \in A$, there is a $b \in B$ such that $b > x$.

(b) There is an $x \in A$ such that, for all $b \in B$, $b > x$.

(c) For all $x, y \in \mathbb{R}$, if $f(x) = f(y)$ then $x = y$.

(d) For all $b \in \mathbb{R}$, there is an $x \in \mathbb{R}$ such that $f(x) = b$.

(e) For all $x, y \in \mathbb{R}$ and all $\varepsilon \in P$, there is a $\delta \in P$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

(f) For all $\varepsilon \in P$, there is a $\delta \in P$ such that, for all $x, y \in \mathbb{R}$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Problem 3 — 2.9

Consider the statement:

No slow learners attend this school.

Which of the following is its negation?

(a) All slow learners attend this school.

(b) All slow learners do not attend this school.

(c) Some slow learners attend this school.

(d) Some slow learners do not attend this school.

(e) No slow learners attend this school.

Problem 4 — 2.11

Suppose I have a penny, a dime and a dollar, and I say, "If you make a true statement, I will give you one of the coins. If you make a false statement, I will give you nothing." What should you say to obtain the best coin?

Problem 5 — 2.16 *

A function $g : \mathbb{R} \to \mathbb{R}$ is *even* if $g(-x) = g(x)$ for all $x \in \mathbb{R}$, or *odd* if $h(-x) = -h(x)$ for all $x \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$.

(a) Prove that there exists a unique pair of functions $g, h : \mathbb{R} \to \mathbb{R}$ such that g is even, h is odd, and $f = g + h$. (**Hint:** Express both $f(x)$ and $f(-x)$ in terms of $g(x)$ and $h(x)$, and solve the resulting system of equations.)

(b) When f is a polynomial function, express g and h as in (a) in terms of the coefficients of f.

Problem 6 — 2.25 *

For $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, show that (a) and (b) below have different meanings.

(a) $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$.

(b) $(\exists \delta > 0)(\forall \varepsilon > 0)(\forall x \in \mathbb{R})[|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon]$.

(**Hint:** Find a function $f : \mathbb{R} \to \mathbb{R}$ and an element $a \in \mathbb{R}$ for which (a) and (b) have different truth values.)

Problem 7 — 2.28 *

(a) Show that the following statement is false:

$$\text{For all } a, x \in \mathbb{R} \text{ there is a unique } y \in \mathbb{R} \text{ such that } x^4y + ay + x = 0$$

(b) Find the set of real numbers a such that the following statement is true:

$$\text{For all } x \in \mathbb{R} \text{ there is a unique } y \in \mathbb{R} \text{ such that } x^4y + ay + x = 0$$

Problem 8 — 2.34 *

For each statement below, decide whether it is true or false. Prove your claim using only properties of the natural numbers.

(a) If $n \in \mathbb{N}$ and $n^2 + (n+1)^2 = (n+2)^2$, then $n = 3$.

(b) For all $n \in \mathbb{N}$, it is false that $(n-1)^3 + n^3 = (n+1)^3$.

Problem 9 — 2.40

See page 48 of the textbook for this problem.

Problem 10 — 2.48 *

Let $P(x)$ be the assertion "x is odd" and let $Q(x)$ be the assertion "x is two times an integer". Determine, with proof, whether the following statements are true or false:

(a) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$.

(b) $(\forall x \in \mathbb{Z})(P(x)) \Rightarrow (\forall x \in \mathbb{Z})(Q(x))$.

Problems 11 and 12 on next page.

Problem 11 — 2.50

Prove the following identities involving complementation of sets.

(a) $(A \cup B)^c = A^c \cap B^c$;

(b) $A \cap [(A \cap B)^c] = A - B$;

(c) $A \cap [(A \cap B^c)^c] = A \cap B$;

(d) $(A \cup B) \cap A^c = B - A$.

Problem 12 *

Let S be a non-empty set of people in a bar. Express the following statement symbolically:

> *There is a person in the bar such that, if that person is drinking, then everyone else in the bar is drinking.*

Prove that it is true.