21-128 problem sheet 1

Solutions to starred (*) exercises are due at the beginning of recitation on

Thursday 10th September 2015

Please submit answers to separate questions on separate sheets of paper.

Problem 1 — 1.8

In the morning section of a calculus course, 2 of the 9 women and 2 of the 10 men receive the grade of A. In the afternoon section, 6 of the 9 women and 9 of the 14 men receive an A. Verify that, in each section, a higher proportion of women than of men receive an A, but that, in the combined course, a lower proportion of women than men receive an A. Explain!

Problem 2 — 1.15

For what conditions on sets A and B does $A - B = B - A$ hold?

Problem 3 — 1.22 *

We have two identical glasses. Glass 1 contains x ounces of wine; glass 2 contains x ounces of water ($x \geq 1$). We remove 1 ounce of wine from glass 1 and add it to glass 2. The wine and water in glass 2 mix uniformly. We now remove 1 ounce of liquid from glass 2 and add it to glass 1. Prove that the amount of water in glass 1 is now the same as the amount of wine in glass 2.

Problem 4 — 1.27

Determine the set of real solutions x to the inequality

$$\left| \frac{x}{x+1} \right| \leq 1$$

Problem 5 — 1.29 *

Let x, y, z be nonnegative real numbers such that $y + z \geq 2$. Prove that

$$(x + y + z)^2 \geq 4x + 4yz$$

Problem 6 — 1.32 *

Assuming only arithmetic (not the quadratic formula or calculus), prove that

$$\{x \in \mathbb{R} : x^2 - 2x - 3 < 0\} = \{x \in \mathbb{R} : -1 < x < 3\}$$

Problem 7 — 1.36 *

Let $S = [3] \times [3]$ (the Cartesian product of $\{1, 2, 3\}$ with itself). Let T be the set of ordered pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \leq 3x + y - 4 \leq 8$. Prove that $S \subseteq T$. Does equality hold?

Problem 8 — 1.42

Let $A = \{\text{January}, \text{February}, \ldots, \text{December}\}$. Given $x \in A$, let $f(x)$ be the number of days in x. Does f define a function from A to $\mathbb{N}$?

Problem 9 — 1.47 *

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by

$$f(a, b) = \frac{(a+1)(a+2b)}{2}$$

(a) Show that the image of f is a subset of $\mathbb{N}$.

(b) Determine exactly which natural numbers are elements of the image of f. (**Hint:** Formulate a hypothesis by trying values.)

Problem 10 — 1.54

Let $S = \{(x, y) \in \mathbb{R}^2 : y \leq x \text{ and } x + 3y \geq 8 \text{ and } x \leq 8\}$.

(a) Graph the set S.

(b) Find the minimum value of $x + y$ such that $(x, y) \in S$. (**Hint:** On the graph from part (a), sketch the level sets of the function f defined by $f(x, y) = x + y$.)

Problem 11 *

Let r be a rational number and let a and b be irrational numbers. Which of the following numbers is necessarily irrational?

$$a + r \quad a + b \quad ar \quad ab \quad a^r \quad r^a \quad a^b$$

Prove your claims, either by proving that the number is irrational or by providing a counterexample. If you claim that a number is irrational, you should prove it.