Functions and cardinality

21-127 sections A and F — TA: Clive Newstead — 29^{th} April 2014

What follows is a list of questions that you might want to try in preparation for my review session on functions on cardinality, which is

Tuesday 6th May 2014 at 7–10pm in Wean Hall 8220

1 Determine which of the following functions are injective and which are surjective:

(a) $f : \mathbb{Z} \to \mathbb{N}$, where $\forall n \in \mathbb{Z}.\ f(n) = |n| + 1$;

(b) $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $\forall (n, k) \in \mathbb{N} \times \mathbb{N}.\ g(n, k) = 2^n \cdot 3^k$;

(c) $h : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, where $\forall A \in \mathcal{P}(\mathbb{N}).\ h(A) = \mathbb{N} \setminus A$;

(d) $k : \mathbb{Z} \to \mathcal{P}(\mathbb{Z})$, where $\forall n \in \mathbb{Z}.\ k(n) = \{n, 1, -1\}$.

Using the above functions, compute the following sets:

$$\text{PreIm}_f(\mathbb{N}), \quad \text{Im}_g(\mathbb{N} \times \{1\}), \quad \text{PreIm}_h(\varnothing), \quad \text{PreIm}_h(\{\varnothing\}), \quad \text{Im}_k(\{-1, 0, 1\})$$

2 The following functions are bijective; find their inverses:

(a) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, where $\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}.\ f(x, y) = (4x - y, y - 3x)$;

(b) $g : [8] \to \mathbb{Z}_8$, where $\forall n \in [8].\ g(n) = [\![3n + 5]\!]$;

3 Define a function $f : \mathbb{Z} \to \mathbb{Z}$ such that, for each $n \in \mathbb{Z}$, $|\text{PreIm}_f(\{n\})| = 2$. Does your function have an inverse? If so, find it; if not, explain why.

4 Let $g : A \to B$ be a function. Under what condition(s) on g are the following statements true?

(a) $\forall b \in B.\ \text{Im}_g(\text{PreIm}_g(\{b\})) = \{b\}$;

(b) $\forall a \in A.\ \text{PreIm}_g(\text{Im}_g(\{a\})) = \{a\}$;

(c) $\exists b \in B.\ \text{PreIm}_g(\{b\}) = \varnothing$.

5 Prove that if $f : [a] \to [b]$ is surjective then $a \geqslant b$.

6 Prove that $f : A \to B$ is injective if and only if $\forall b \in B.\ |\text{PreIm}_f(\{b\})| \leqslant 1$.

7 Two sets A and B are defined by:

$$A = \{n \in \mathbb{Z} : n \equiv 2 \bmod 3\} \quad \text{and} \quad B = \{n \in \mathbb{Z} : n \equiv 0 \bmod 7\}$$

Find a bijection from A to B and give an expression for its inverse.

8 Find a subset $A \subseteq \mathbb{R}$ for which the function $f : A \to \mathbb{R}$ given by $f(x) = x^2 - 3x + 2$ is injective. (Bonus points if A is *maximal*, i.e. if $A \subsetneqq B \subseteq \mathbb{R}$ then $\widehat{f} : B \to \mathbb{R}$ given by $\widehat{f}(x) = x^2 - 3x + 2$ is *not* injective.)

More advanced questions available upon request.