Optional assignment

21-120 sections A and B   TA: Clive Newstead   Mailbox: Wean Hall 6113

These questions are designed to be tricky and to make you think about stuff. If you want feedback on your answers, please return them to me by 5pm on Thursday (26th September 2013); you can collect them during my office hours on Friday (11–12am, Wean Hall 8205).

Question 1: Domains

Find the domain of the function $f(x) = \sqrt{\ln(\sin(\frac{1}{x}))}$.

Question 2: Limits

Calculate the limit $\lim\limits_{\theta \to 0} \dfrac{\sin 2\theta - 2\sin\theta - \sin^3\theta}{\theta^2}$.

Question 3: Continuity

A real number x is *rational* if it can be written as a ratio of two integers, i.e. if $x = \dfrac{m}{n}$ for some integers m and n; otherwise it is *irrational*. It is known that between any two rational numbers there is an irrational number, and between any two irrational numbers there is a rational number.

Define
$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Find the points of continuity of each of the following functions:

$$f(x), \quad xf(x), \quad \cos(x + \frac{\pi}{3})f(x), \quad f(f(x))$$

Question 4: Intermediate value theorem

Use the intermediate value theorem to prove that there are infinitely many solutions to the equation $x = \sin\frac{1}{x}$ in the interval $0 < x \leqslant 1$.

Question 5: ε-δ proofs

Prove from the ε-δ definition of a limit that if $g(x)$ is a bounded function and $f(x) = (x - a)g(x)$ then $\lim\limits_{x \to a} f(x)$ exists. What is the limit?

Question 6: Derivatives

(a) Find y' when $y = x^x$, leaving your answer in terms of x. Use your answer to find the equation of the tangent line to the curve $y = x^x$ when $x = 3$.

(b) Find z' when $z = x^{x^x}$, leaving your answer in terms of x. Use your answer to find the equation of the normal line to the curve $z = x^{x^x}$ when $x = 2$.

Question 7: More derivatives

Find a function f satisfying the following conditions:

$$f''(x) + 4f(x) = 0, \quad f(0) = 3, \quad f'(0) = 0$$

Question 8: Implicit differentiation

Prove that a line intersecting a circle does so at a right-angle if and only if it passes through the circle's centre.

Question 9: Related rates

A spherical blob of ink of radius 1 cm is dropped into a tank full of water and begins to diffuse uniformly. At the onset, the radius is increasing at 2 cm per second. The volume of the blob of ink increases at a constant rate. How fast is the blob's radius increasing when its volume reaches 1 m^3?

Hints

Q2. Look out for useful trig identities.

Q6. Use logarithms and differentiate implicitly.

Q7. Think trig; use the chain rule.