

# Integration by substitution

I've thrown together this step-by-step guide to integration by substitution as a response to a few questions I've been asked in recitation and office hours. If you notice any mistakes or have any questions please throw them in my direction by sending an email to [cnewstead@cmu.edu](mailto:cnewstead@cmu.edu).

—Clive Newstead

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## 1 When to substitute

There are two types of integration by substitution problem:

(a) Integrals of the form  $\int_a^b f(g(x))g'(x) dx$ . In this case we'd like to substitute  $u = g(x)$  to simplify the integrand.

(b) Integrals of the form  $\int_a^b f(x) dx$ , when  $f$  is some weird function whose antiderivative we don't know. In this case we'd like to substitute  $x = h(u)$  for some cunningly-chosen function  $h$  to transform the integrand into something whose antiderivative we know.

This is only a rule of thumb: sometimes we might want to make a substitution of the form  $u = g(x)$  even when our integral is of the second form.

## 2 Summary: the steps

**Step 1: Choose a substitution to make.**

**Step 2: Find  $dx$  in terms of  $du$ .**

**Step 3: Look at the limits.** (*Definite integrals only.*)

**Step 4: Make the substitution.**

**Step 5: Compute the new integral.**

**Step 6: Substitute back again.** (*Indefinite integrals only.*)

## 3 Explanation of the steps

**Step 1: Choose a substitution to make.** This might be  $u = g(x)$  or  $x = h(u)$  (or maybe even  $g(x) = h(u)$ ) according to the problem in hand. The hardest part when integrating by substitution is finding the right substitution to make: this comes with (lots and lots and **lots** of) practice.

**Step 2: Find  $dx$  in terms of  $du$ .** This can be done by differentiating the variable you want to substitute. In the case  $u = g(x)$  we get  $du = g'(x) dx$ , so  $dx = \frac{1}{g'(x)} du$ . In the case  $x = h(u)$  we simply get  $dx = h'(u) du$ .

**Step 3: Look at the limits.** (*Definite integrals only.*) If you substitute  $u = g(x)$  then the limits will be  $g(a)$  and  $g(b)$ ; if  $x = h(u)$  then the limits will be  $h^{-1}(a)$  and  $h^{-1}(b)$ .

In order to keep track of the limits it might help to write the integral sign as  $\int_{x=a}^{x=b}$ . This really drills home that the limits refer to values of  $x$  and not of the new variable, and it helps you keep track of whether you need to take the inverse or not.

For instance, if your limit is  $x = a$  and your substitution is  $u = g(x)$  then we just write  $x = a$  in  $u = g(x)$  to get  $u = g(a)$ . Likewise, if the substitution is  $x = h(u)$  then we need to solve  $a = h(u)$ , so we get  $u = h^{-1}(a)$ .

**Step 4: Make the substitution.** We now have the information we need in order to transform the integral. There should be no traces of  $x$  left: they should all be replaced by  $u$  in one way or another. Don't forget to substitute for  $dx$  and to change the limits (if there are any)!

**Step 5: Compute the new integral.** There wouldn't be much point in making the substitution if we didn't compute the integral. It sounds silly, but people often forget! If your integral had limits, you can plug them in to obtain a numerical answer using the Fundamental Theorem of Calculus.

**Step 6: Substitute back again.** (*Indefinite integrals only.*) Now you should have something that looks like  $F(u) + C$ ; to finish the job we need to write this in terms of  $x$ . If you substituted  $u = g(x)$  then the final answer will be  $F(g(x)) + C$ . If you substituted  $x = h(u)$  then the final answer will be  $F(h^{-1}(u)) + C$ .

What follows are two worked examples to see these steps in action.

## 4 Worked example 1: $\int_2^6 x \sin(x^2 + 9) dx$

**Step 1: Choose a substitution to make.** The first thing to do is to scour for ‘functions of functions’. There’s a fairly obvious one:  $\sin(x^2 + 9)$  is the function  $\sin$  applied to  $x^2 + 9$ . So we’ll substitute

$$u = x^2 + 9$$

This is likely to work:  $x$  multiplies  $\sin(x^2 + 9)$  in the integral, and the derivative of  $x^2 + 9$  is  $2x$ , which is a constant multiple of  $x$ .

**Step 2: Find  $dx$  in terms of  $du$ .** If  $u = x^2 + 9$  then  $du = 2x dx$ , so

$$dx = \frac{du}{2x}$$

**Step 3: Look at the limits.** Our limits are  $x = 2 \nearrow 6$ . If  $x = 2$  then  $u = 2^2 + 9 = 13$ , and if  $x = 6$  then  $u = 6^2 + 9 = 45$ . So the limits of the new integral are

$$u = 13 \nearrow 45$$

**Step 4: Make the substitution.** Substituting all the new values in we have

$$\int_{x=2}^{x=6} x \sin(x^2 + 9) dx = \int_{u=13}^{u=45} x \sin(u) \frac{1}{2x} du = \int_{u=13}^{u=45} \frac{1}{2} \sin u du$$

I’ve used colours to indicate where the different parts come from: red for the limits, blue for the direct substitution  $u = g(x)$  and purple for the  $dx$  substitution.

**Step 5: Compute the new integral.** We now have something we can compute using the Fundamental Theorem of Calculus:  $\frac{d}{dx}(-\cos x) = \sin x$ , and so

$$\int_{13}^{45} \frac{1}{2} \sin(u) du = \left[ \frac{1}{2}(-\cos u) \right]_{13}^{45} = \frac{1}{2}(-\cos(45)) - \frac{1}{2}(-\cos(13)) = \boxed{\frac{1}{2}(\cos(13) - \cos(45))}$$

## 5 Worked example 2: $\int \frac{1}{\sqrt{9 - x^2}} dx$

**Step 1: Choose a substitution to make.** Like last time, we look for ‘functions of functions’. Well there’s certainly one of those: the function  $\frac{1}{\sqrt{\dots}}$  applied to  $9 - x^2$ . You might be tempted to substitute  $u = 9 - x^2$ , but this won’t work: the derivative of this is  $-2x$ , and there are no common multiples of  $x$  kicking around.

In this case a more clever idea is needed. Look at the identity  $\sin^2 \theta + \cos^2 \theta = 1$ . Multiplying by 9 we get  $9 \sin^2 \theta + 9 \cos^2 \theta = 9$ , and rearranging gives  $9 - 9 \sin^2 \theta = 9 \cos^2 \theta$ . Taking square roots then gives  $\sqrt{9 - (3 \sin \theta)^2} = 3 \cos \theta$ . So let’s substitute...

$$x = 3 \sin \theta$$

**Step 2: Find  $dx$  in terms of  $d\theta$ .** Differentiating gives

$$dx = 3 \cos \theta d\theta$$

**Step 3: Look at the limits.** This is an indefinite integral: there are no limits!

**Step 4: Make the substitution.** Substituting  $x = 3 \sin \theta$  and  $dx = 3 \cos \theta d\theta$  gives

$$\int \frac{1}{\sqrt{9-x^2}} dx = \int \frac{1}{\sqrt{9-(3 \sin \theta)^2}} \cdot 3 \cos \theta d\theta = \int \frac{1}{3 \cos \theta} \cdot 3 \cos \theta d\theta = \int 1 d\theta$$

Magic!

**Step 5: Compute the new integral.** Our integral is now very easy to evaluate:

$$\int 1 d\theta = \theta + C$$

for some constant  $C$ .

**Step 6: Substitute back again.** We substituted  $x = 3 \sin \theta$ , so  $\theta = \sin^{-1} \left( \frac{x}{3} \right)$ , and hence

$$\int \frac{1}{9-x^2} dx = \boxed{\sin^{-1} \left( \frac{x}{3} \right) + C}$$

for some constant  $C$

## 6 Optional exercises

**DO NOT GIVE THESE TO YOUR TA!** (unless I'm your TA) If you're in Section A or B then feel free to submit them in recitation. If you're in another section you'd like feedback on your solutions then please send me an email (details on the first page).

1.  $\int_1^5 x \cos(\pi x^2) dx$

2.  $\int_4^{10} (x+2) \cos[(x+1)(x+3)] dx$

3.  $\int_0^\pi \cos \theta \cos(\sin \theta) d\theta$

4.  $\int e^{4x} \sin(\pi + e^{4x}) dx$

5.  $\int_a^b \frac{1}{x \ln x} dx$ , for arbitrary constants  $a, b$

6.  $\int \frac{f'(x)}{(f(x) - 2013)^2} dx$ , for an arbitrary function  $f$

7.  $\int \sqrt{1-4x^2} dx$ , using the substitution  $x = \frac{1}{2} \sin \theta$

[*Bonus*: why did we choose this substitution?]

8.  $\int \frac{1}{4+x^2} dx$

[*Hint*: consider the identity  $1 + \tan^2 \theta = \sec^2 \theta$  and follow Example 2 above.]

9.  $\int \frac{e^t}{25 + e^{2t}} dt$

[*Hint*: more than one substitution may be necessary.]