15-151 Homework 9

Please submit in class at 8:00am on Friday 4th August

Exercises

1. Let X be a finite set with $|X| = n \geq 1$.
   (a) Let $a \in X$. Prove that $|X \setminus \{a\}| = n - 1$. **[6 points]**
   (b) Does part (a) remain true if we drop the requirement that $a \in X$? Why or why not? **[2 points]**

2. (a) Count the set of six-card hands from a standard 52-card deck that contain at least one card of each suit. **[4 points]**
   (b) Fix $n \in \mathbb{N}$. Prove that the number of surjections $[n] \to [3]$ is $3^n - 3 \cdot 2^n + 3$. **[6 points]**

3. Prove by double counting that $\displaystyle\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = 2^{n-m}\binom{n}{m}$, where m, n are fixed natural numbers such that $m \leq n$. **[4 points]**

Project milestone

Please submit your work for the project milestone separately from your homework.

- State and prove the main result of your project. **[3 points]**
- Answer the prescribed questions for your project. **[6 points]**

Optional problems

Homeworks 9 and 10 will contain additional optional problems totalling 35 points between them (14 points on Homework 9, and 21 points on Homework 10). You do not need to attempt these problems. If you do, then your score out of 35 will replace your lowest homework score from Homeworks 1 through 8.

Opt 1. Let $m, n \in \mathbb{N}$. Prove that there is a bijection $[mn] \to [m] \times [n]$. **[7 points]**

Opt 2. Prove by double counting that $\displaystyle\sum_{i=1}^{n} \binom{i-1}{2}(n-i) = \binom{n}{4}$ for all $n \in \mathbb{N}$. **[7 points]**