15-151 Homework 7

Please submit in class at 8:00am on Thursday 27th July

Exercises

1. Let p and q be distinct prime integers and let $k, \ell, m, n \in \mathbb{N}$. Prove that
$$\gcd(p^k q^\ell, p^m q^n) = p^{\min(k,m)} q^{\min(\ell,n)}$$
[7 points]

2. Let $a, p \in \mathbb{Z}$ and suppose that p is prime. Prove that $a \perp p$ if and only if $p \nmid a$. **[7 points]**

3. Let $a, b \in \mathbb{Z}$ and let $d = \gcd(a, b)$. Prove that $\frac{a}{d} \perp \frac{b}{d}$. **[7 points]**

4. Let $n \in \mathbb{Z}$ with $n > 2$. Prove that the set
$$\{k \in \mathbb{Z} \mid n < k < n!\}$$
contains a prime number, where $n! = 1 \times 2 \times \cdots \times n$ is the *factorial* of n. **[7 points]**
Hint: consider the prime factors of $n! - 1$.

5. Let $n \in \mathbb{Z}$ with $n > 2$. Prove that $n = m^2$ for some $m \in \mathbb{Z}$ if and only if the multiplicity of each prime in the prime factorisation of n is even. **[7 points]**