15-151 Homework 4

Please submit in class at 8:00am on Tuesday 18th July

Exercises

1. (a) Prove that $(p \vee q) \Rightarrow r$ is logically equivalent to $(p \Rightarrow r) \wedge (q \Rightarrow r)$. **[3 points]**

   (b) Draw a truth table demonstrating that $p \vee (\neg p)$ is true, no matter whether p is true or false. **[1 points]**

   (c) Suppose you know that $p \Rightarrow r$ and $(\neg p) \Rightarrow r$ are true. Explain how parts (a) and (b) imply that r is true. **[1 points]**

2. Define a new logical operator $\uparrow$ by letting $p \uparrow q$ be true if p and q are not both true, and false otherwise.

   (a) Draw the truth table for $p \uparrow q$. **[1 points]**

   (b) Prove that $\neg p \equiv p \uparrow p$. **[1 points]**

   (c) Prove that $p \wedge q \equiv (p \uparrow q) \uparrow (p \uparrow q)$. **[2 points]**

   (d) Express $p \vee q$ in terms of the variables p and q and the logical operator $\uparrow$, and prove that your expression is valid. **[3 points]**

   (e) Express $p \Rightarrow q$ in terms of the variables p and q and the logical operator $\uparrow$, and prove that your expression is valid. **[3 points]**

3. The *unique existential quantifier* '$\exists!$' is defined by saying $\exists! x \in X,\ p(x)$ is true if there is exactly one value of $x \in X$ such that $p(x)$ is true. More precisely, it can be defined in terms of $\exists$ and $\forall$ by saying

   $$\exists! x \in X,\ p(x) \quad \equiv \quad \exists x \in X,\ (p(x) \wedge \forall y \in X,\ (p(y) \Rightarrow y = x))$$

   Using this definition, write out $\neg \exists! x \in X,\ p(x)$ as a maximally negated logical formula containing only the quantifiers $\forall$ and $\exists$.

   [5 points]

4. Write out the following sets in set-builder notation, without using English words; for example, the set of all even integers is $\{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z},\ n = 2k\}$.

   (a) The set of all odd integers; **[3 points]**

   (b) The set of all irrational numbers; **[3 points]**

   (c) The set of all natural numbers which can be expressed as $3u + 7v$ for some natural numbers u and v. **[4 points]**

Please also complete the task on the next page

Other tasks

5. Fill out the two-week questionnaire located at the following URL (which will be activated after the LaTeX project submission deadline): `https://goo.gl/forms/o7zFaI2qYaTaeKQE3`

[5 points]

Optional but recommended tasks (not for credit)

6. There are sixteen possible truth tables for propositional formulae containing two propositional variables. Prove that all propositional formulae containing two propositional variables is equivalent to one containing only the logical operator $\uparrow$ defined in Question 2. Deduce that all propositional formulae, no matter how many propositional variables they contain, can be expressed using only the logical operator $\uparrow$.

7. Another way of defining the unique existential quantifier is by letting

$$\exists! x \in X,\, p(x) \quad \equiv \quad [\exists x \in X,\, p(x)] \wedge [\forall y, z \in X,\, (p(x) \wedge p(y)) \Rightarrow x = y]$$

   Explain how this definition, and the definition given in Question 3, inform us of two ways in which we can prove statements of the form $\exists! x \in X,\, p(x)$.

8. Let p and q be propositional variables, and let $C(p,q)$ be some logical formula involving only the variables p and q. Suppose that $C(p,q)$ satisfies the following conditions:

   (i) $C(p,q) \Rightarrow p$ and $C(p,q) \Rightarrow q$;

   (ii) If X is any propositional formula such that $X \Rightarrow p$ and $X \Rightarrow q$, then $X \Rightarrow C(p,q)$.

   Prove that $C(p,q) \equiv p \wedge q$.