15-151 Homework 3

Please submit in class at 8:00am on Thursday 13th July

Exercises

1. Consider the sequence $(a_n)_{n \geq 0}$ defined by

$$a_0 = 2, \quad a_1 = 3 \quad \text{and} \quad a_{n+2} = 3a_{n+1} - 2a_n \text{ for all } n \in \mathbb{N}$$

   (a) Evaluate the terms a_2, a_3, a_4 and a_5. **[1 points]**

   (b) Find an expression for a general term a_n of the sequence in terms of n only. **[2 points]**

   (c) Prove by induction that your expression is valid. **[7 points]**

2. (a) Which natural numbers less than fifteen can be expressed as $3u + 7v$ for some $u, v \in \mathbb{N}$? Prove your claims. **[1 points]**

   (b) Which natural numbers less than fifteen *cannot* be expressed as $3u + 7v$ for some $u, v \in \mathbb{N}$? Prove your claims. **[2 points]**

   (c) Prove that every natural number greater than or equal to fifteen can be expressed as $3u+7v$ for some $u, v \in \mathbb{N}$. [*Hint: use strong induction.*] **[7 points]**

3. Fix $c, d \in \mathbb{R}$ and suppose that a sequence $(x_n)_{n \geq 0}$ satisfies

$$x_0 = 0, \quad x_1 = 1 \quad \text{and} \quad x_{n+2} = cx_{n+1} + dx_n \text{ for all } n \in \mathbb{N}$$

   Suppose furthermore that $c^2 + 4d > 0$, and let α and β be the (distinct, real) roots of the quadratic $x^2 - cx - d$. [This means that $x^2 - cx - d = (x - \alpha)(x - \beta)$ for all $x \in \mathbb{R}$.]

   (a) Find real numbers A and B, defined in terms of α and β, such that $x_0 = A + B$ and $x_1 = A\alpha + B\beta$. **[2 points]**

   (b) Using the values of A and B you found in part (a), prove by induction that $x_n = A\alpha^n + B\beta^n$ for all $n \in \mathbb{N}$. **[6 points]**

   (c) The *Fibonacci sequence* is the sequence $(f_n)_{n \geq 0}$ defined by

$$f_0 = 0, \quad f_1 = 1 \quad \text{and} \quad f_{n+2} = f_{n+1} + f_n \text{ for all } n \in \mathbb{N}$$

   Using your answers to parts (a) and (b) to prove that

$$f_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \text{ for all } n \in \mathbb{N}$$

   where $\varphi = \dfrac{1 + \sqrt{5}}{2}$ and $\psi = \dfrac{1 - \sqrt{5}}{2}$. **[2 points]**

More tasks on the next page

Other tasks

4. Typeset your answer to (at least) one of the previous questions using LaTeX. Upload the `.tex` file to the Homework 2 page on Canvas. Print out the resulting `.pdf` document and submit it along with your answers to the other questions. **[5 points]**

Optional but recommended tasks (not for credit)

5. Typeset all of your homework solutions using LaTeX.

6. The number φ in Q3(c) is known as the *golden ratio*. Despite its simple definition as being the greatest of the two roots of the quadratic $x^2 - x - 1$, approximations to it appear all over the place in nature. This has fascinated people over the ages, who have had reactions ranging from, 'Oh, that's interesting!', to outright mysticism. Read about some of the occurrences of φ in the real world.

7. Suppose that the first two terms of the sequence in Question 3 had been defined by $x_0 = a$ and $x_1 = b$, where a and b are two more fixed real numbers. Repeat Q3(a) in this case, and show that Q3(b) remains true. Apply your result to the case where $a = 2$, $b = 3$, $c = 3$ and $d = -2$ and verify that your answer agrees with that of Q1(b).

8. Suppose in Question 3 that we had $c^2 + 4d = 0$, so that $\alpha = \beta$. What goes wrong? Show that, in this case, we have $x_n = (A + Bn)\alpha^n$ for all $n \in \mathbb{N}$, where A and B are real numbers, defined in terms of α, that you should find.