21-624

Descriptive Set Theory

taught in Spring 2025 by

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Descriptive Set Theory
Lecture 1

Textbook: Kechris' Classical Descriptive Set Theory

Homework policy:

- On (roughly) biweekly schedule
- Some are (much) harder than others
- I expect conversation(s) from everybody
- Final presentation on an advanced topic

Expected background:

- Familiarity with the language of real analysis and topology
- Basic set theory: countable vs uncountable transfinite induction, etc.
- Groups and graphs will probably show up

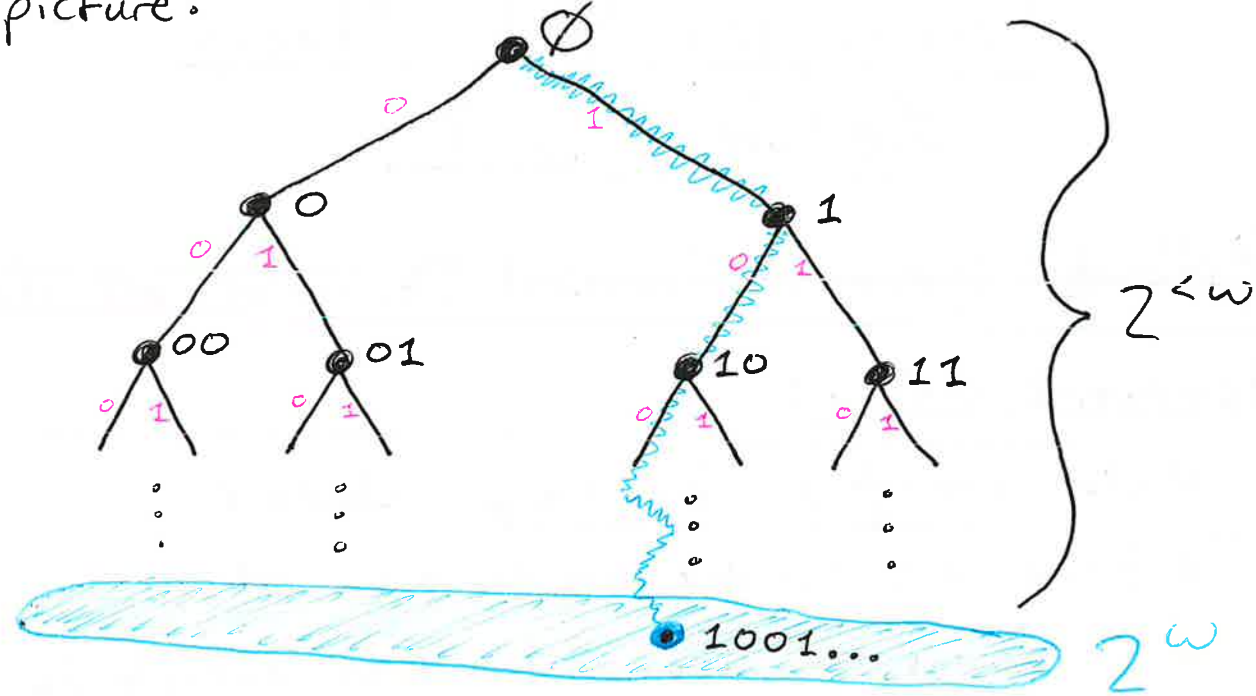
Def: $2 = \{0, 1\}$

$\omega = \mathbb{N} = \{0, 1, 2, \ldots\}$

$2^\omega = {}^\omega 2 = \{\text{functions } \omega \to 2\}$
$= \{\mathbb{N}\text{-indexed binary strings}\}$

$2^{<\omega} = \{\text{finite binary strings}\}$

② The picture:



<u>Remarks</u>: $|2^{\omega}| = 2^{\aleph_0} = |\mathbb{R}|$

$|2^{<\omega}| = \aleph_0 = |\mathbb{N}|$

<u>Thm</u> (Cantor): Suppose that $C \subseteq \mathbb{R}$ is closed.

Exactly one:

$\boxed{\mathrm{I}}$ C is countable

$\boxed{\mathrm{II}}$ There is a (continuous) injection $f: 2^{\omega} \to C$.

So... "CH holds for closed sets."

<u>Note:</u> Since 2^{ω} is unctbl, $\boxed{\mathrm{I}}$ & $\boxed{\mathrm{II}}$ are exclusive. So it suffices to show that $\boxed{\mathrm{II}}$ holds whenever $C \subseteq \mathbb{R}$ is closed and unctbl.

③ Key observation: The complete binary tree $2^{<\omega}$ "splits" into two smaller copies of itself. We shall emulate this behavior in $\mathcal{C}$.

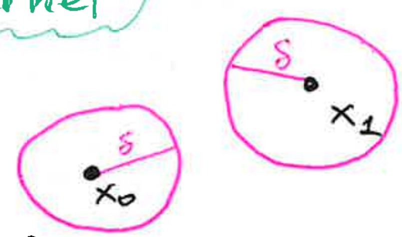
Splitting Lemma: Suppose that $C \subseteq \mathbb{R}$ is closed and unctbl, and $\varepsilon > 0$. Then $\exists C_0, C_1 \subseteq C$ s.t.

- Each C_i is closed and unctbl
- $C_0 \cap C_1 = \emptyset$
- $\operatorname{diam}(C_i) \leq \varepsilon$ $\{x, y \in C_i \Rightarrow d(x,y) \leq \varepsilon\}$

pf (S.L.): Put $\mathcal{U} = \bigcup\{(q,r) : q, r \in \mathbb{Q} \text{ and } C \cap (q,r) \text{ ctbl}\}$.

So $C \cap \mathcal{U}$ is ctbl (in fact, $\mathcal{U}$ is $\boxed{\text{max'l}}$ open with $C \cap \mathcal{U}$ ctbl)

Put $C^* = C \setminus \mathcal{U}$ {the perfect kernel}



C^* is unctbl, so in particular has two distinct elements $x_0, x_1 \in C^*$.

Pick δ smaller than $\varepsilon/2$ and $d(x_0,x_1)/3$.

Then $C_i = C \cap [x_i - \delta, x_i + \delta]$ works. ■ (S.L.)

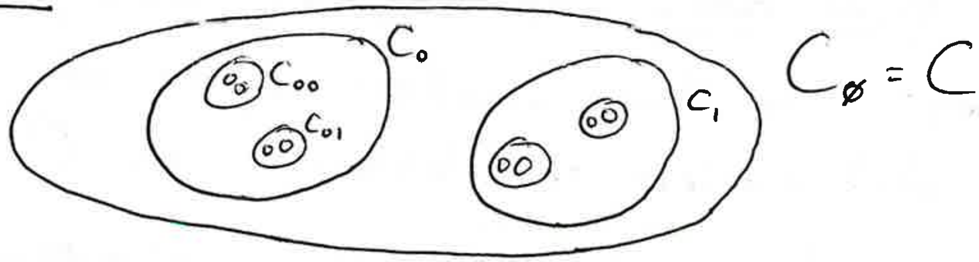
pf (Thm): Suppose that $C \subseteq \mathbb{R}$ is closed, unctbl.

Iterate S.L., recursively defining $C_s \subseteq \mathbb{R}$ for each $s \in 2^{<\omega}$ as follows: {closed, unctbl}

- $C_\emptyset = C$
- Given C_s, $C_{s^\frown 0}$ and $C_{s^\frown 1}$ are disjoint closed unctbl subsets of C_s with

$$\operatorname{diam}(C_{s^\frown i}) \leq \frac{1}{\operatorname{len}(s)+1}.$$

④ pf (Thm, cont.)



For each $x \in 2^\omega$, the set $\bigcap_{n \in \omega} C_{x \restriction n}$ is a singleton by completeness.

Let $f(x)$ be the unique element of this set.

Then $f: 2^\omega \to C$ is the desired injection!

■ (Thm)

What special information about $\mathbb{R}$ (as a topological space) did we use? Not very much!

① $\mathbb{R}$ is separable, i.e., it has a ctbl dense set.

- Used in construction of C^* in S.L.

② $\mathbb{R}$ admits a complete metric, i.e., Cauchy seqs converge.

- Used in definition of f.

<u>Def</u>: A topological space is <u>Polish</u> if it is separable and admits a compatible complete metric. {open metric balls generate topology}

<u>Thm</u> (Cantor, redux): If X is an unctbl Polish space, then 2^ω (continuously) injects into X.

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Lecture 2

Trees and sequence spaces

Today, fix some non-$\emptyset$ (wellorderable) sets A and B.

Def: $A^\omega = \{\alpha : \omega \to A\} = \{\omega\text{-indexed sequences from } A\}$

$$A^{<\omega} = \bigcup_{n\in\omega} A^n = \{\text{finite sequences from } A\}$$

Fix any sequence $(\delta_n)_{n\in\omega}$ of positive real numbers decreasing to 0. This induces an ultrametric on A^ω: $d(\alpha,\beta) = \delta_n$ iff α and β first disagree at index n.

Recall: An ultrametric is a metric with a strengthened triangle inequality: $d(x,z) \leq \max\{d(x,y), d(y,z)\}$.
"All triangles are isoceles."

Remark: The above metric on A^ω is complete.

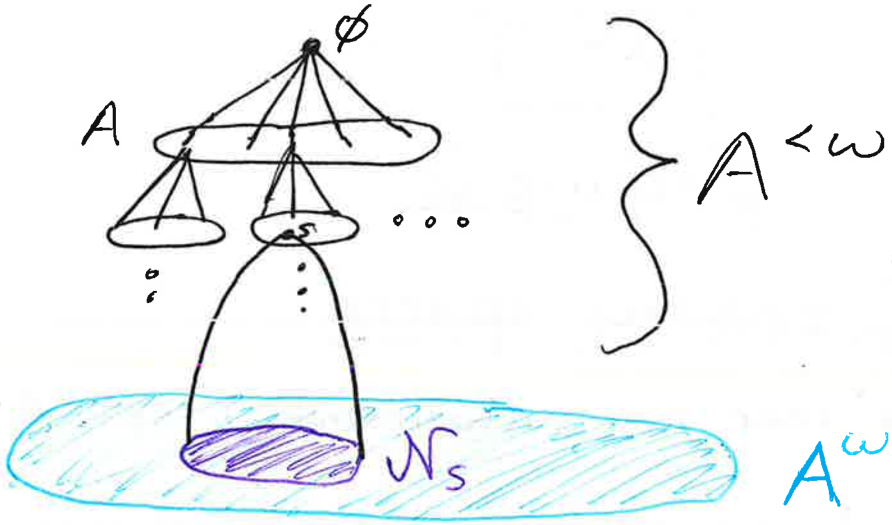
What is the induced topology? We need to understand the δ_n-ball about any $\alpha \in A^\omega$.

Note: $d(\alpha,\beta) < \delta_n$ iff $\alpha \upharpoonright (n+1) = \beta \upharpoonright (n+1)$.

So open balls have the form

$$\mathcal{N}_s = \{\alpha \in A^\omega : s \sqsubseteq \alpha\}$$

for some $s \in A^{<\omega}$.

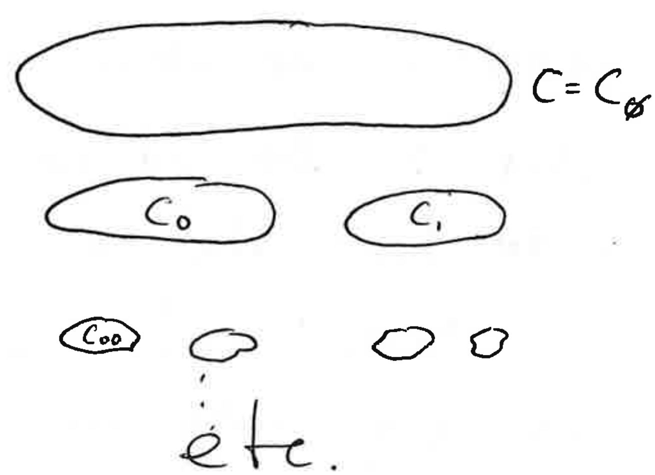


<u>Remark</u>: This topology on A^ω coincides with the (finite-support) product of ω-many copies of (A, discrete topology).

<u>Exercise</u>:
- A^ω is compact iff A is finite
- A^ω is Polish iff A is ctbl.

<u>Remark</u>: It should now be clear why the injection $f: 2^\omega \to C$ from last time is continuous:

If $\alpha, \beta \in 2^\omega$ are very close, they must have a long initial segment $s \in 2^{<\omega}$ in common. Then both $f(\alpha), f(\beta) \in C_s$ and hence they are close.



③ **Def:** 2^ω is called <u>Cantor space</u>.
ω^ω is called <u>Baire space</u>.

Def: $T \subseteq A^{<\omega}$ is called a <u>tree</u> if it is stable under taking initial segments:
$$(t \in T \text{ and } s \sqsubseteq t) \Rightarrow s \in T.$$

Def: Given a tree $T \subseteq A^{<\omega}$, its set of <u>branches</u> is $[T] = \{\alpha \in A^\omega : \forall n \in \omega \ \alpha \restriction n \in T\}$.

Def: A tree $T \subseteq A^{<\omega}$ is <u>pruned</u> if it has no dead ends: $t \in T \Rightarrow \exists a \in A \ t^\frown a \in T$.

Prop: $C \subseteq A^\omega$ is closed iff there is a pruned tree $T \subseteq A^{<\omega}$ with $C = [T]$

pf: Ex. ■

Def: Given trees $S \subseteq A^{<\omega}$ and $T \subseteq B^{<\omega}$, we say a function $f : S \to T$ is <u>monotone</u> if it respects the tree order. I.e.,
$$s_0 \sqsubseteq s_1 \Rightarrow f(s_0) \sqsubseteq f(s_1).$$

Def: For such monotone $f : S \to T$, define a partial function $\varphi_f = \lim f : [S] \longrightarrow [T]$ by $\varphi_f : \alpha \longmapsto \bigcup_{n \in \omega} f(\alpha \restriction n)$.

④ What is the domain of φ_f?

It is $\{\alpha \in [S] : \lim_{n\to\infty} \mathrm{len}(f(\alpha \restriction n)) = \infty\}$

<u>Ex</u>: This set is G_δ in $[S]$.

ctbl intersection of open sets

<u>Prop</u>: Given monotone $f : S \to T$, the partial function $\varphi_f = \lim f$ is continuous on its domain. Conversely, any continuous (total, for convenience) function $\varphi : [S] \to [T]$ has the form $\varphi = \lim f$ for some monotone $f : S \to T$.

<u>pf</u>: $\Rightarrow$: ☺

$\Leftarrow$: Given φ, define $f : S \to T$ by

$f : s \mapsto$ longest t of length $\leq \mathrm{len}(s)$ s.t.

$$\varphi[\mathcal{N}_s \cap [S]] \subseteq \mathcal{N}_t. \quad \blacksquare$$

<u>Prop</u>: Suppose that $C \subseteq D$ are both non-$\emptyset$ closed subsets of A^ω. Then there is a cont. surj $\varphi : D \to C$ s.t. $\forall \alpha \in C$ $\varphi : \alpha \mapsto \alpha$.

<u>pf</u>: Fix pruned trees $S \subseteq T$ with $C = [S]$ and $D = [T]$. Recursively define monotone $f : T \to S$ by "folding left if needed while fixing $s \in S$." Declare $f : \emptyset \mapsto \emptyset$.

$$f : t^\frown a \mapsto \begin{cases} t^\frown a & \text{if } t^\frown a \in S \\ f(t)^\frown b & \text{with } b \text{ least s.t. } f(t)^\frown b \in S \text{ otherwise.} \end{cases}$$

we can wellorder

Put $\varphi = \lim f$. $\blacksquare$

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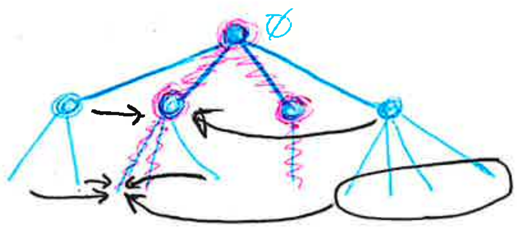
Lecture 3

Last time: Given A discrete, $C \subseteq D$ non-$\emptyset$ closed subsets of A^ω, then C is a retract of D.

Promised picture:



$[S] = C$

$[T] = D$

Remark: The ubiquity of retracts is very special. Think about $C = \{0,1\}$, $D = [0,1]$ as subsets of $\mathbb{R}$.

Recall: If (X_i, τ_i) is an I-indexed sequence of topological spaces, the (finite-support) product topology on $\prod_{i \in I} X_i$ is the coarsest topology rendering every projection $\pi_j : \begin{array}{l} \prod_i X_i \to X_j \\ (x_i)_{i \in I} \mapsto x_j \end{array}$ continuous.

An explicit base is given by

$$\left\{ \mathcal{O}_f : \begin{array}{l} f : i \mapsto \mathcal{U}_i \in \tau_i \text{ and} \\ f(i) = X_i \text{ cofinitely often} \end{array} \right\}$$

where $\mathcal{O}_f = \{ (x_i) \in \prod_i X_i : x_i \in f(i) \}$.

Today: A crash course on compactness.

"Every open cover admits a finite subcover."

②

Facts:

ⓐ (X Hausdorff and $K \subseteq X$ compact) $\Rightarrow$ K closed

ⓑ (X compact and $K \subseteq X$ closed) $\Rightarrow$ K compact

ⓒ Compactness is stable under:
- finite unions
- fin-supp products
- continuous images

ⓓ If K is compact and Y is Hausdorff, then any continuous injection $f : K \hookrightarrow Y$ is in fact a homeomorphism $K \cong f[K]$.

ⓔ Compact metrizable spaces are automatically separable and sequentially compact. Also, any compatible metric is automatically complete.

Let's find them all (in two ways!)

Thm: Every separable metrizable space is homeomorphic to a subset of $[0,1]^\omega$. (w/subspace top)

pf: Fix such a space X, and let $(y_n)_{n\in\omega}$ enumerate a dense subset of X. Fix a compatible metric d on X bounded by 1.

> How? Given any compatible metric d' on X, we can tweak it:
> $$d : (x,y) \longmapsto \min\{d'(x,y), 1\}.$$

pf (Thm, cont.): Define $f : X \to [0,1]^\omega$ by
$$f(x) : k \mapsto d(x, y_k).$$

Put $Z = f[X]$.

Claim 1: f is injective (thus bijective onto Z).

pf(C1): Given $x \neq x'$ in X, find $k \in \omega$ so that
$d(x, y_k) < \frac{1}{3} d(x, x')$.

Then $f(x)(k) \neq f(x')(k)$. $\blacksquare$(C1)

Claim 2: f-preimages of open sets are open.

pf(C2): It suffices to show for all $k \in \omega$ and open $\mathcal{U} \subseteq [0,1]$ that $f^{-1}\left([0,1]^k \times \mathcal{U} \times [0,1]^\omega\right) \subseteq X$ is open. But it's $\{x \in X : d(x, y_k) \in \mathcal{U}\}$. $\blacksquare$(C2)

Claim 3: f-images of open sets are open in Z.

pf(C3): It suffices to show that each basic open ball $B(y_k ; q) = \{x \in X : d(x, y_k) < q\}$ has open image in Z. We check
$$f[B(y_k ; q)] = Z \cap [0,1]^k \times [0,q) \times [0,1]^\omega$$
which is open in Z as desired. $\blacksquare$(C3)

The three claims ensure that f is our desired homeomorphism between X and Z.

$\blacksquare$(Thm)

Cor: Every compact metrizable space is homeomorphic to a closed subset of $[0,1]^\omega$.

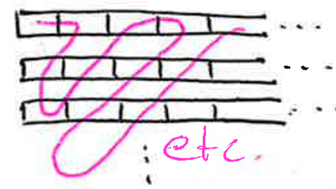
④

Prop: $[0,1]$ is a continuous image of 2^ω.

pf: Binary expansion: $\alpha \overset{?}{\mapsto} \sum_n \alpha(n) 2^{-n-1}$. ∎(Prop)

Prop: $[0,1]^\omega$ is a continuous image of 2^ω.

pf: The prev. prop implies that $[0,1]^\omega$ is a cont. image of $(2^\omega)^\omega$.



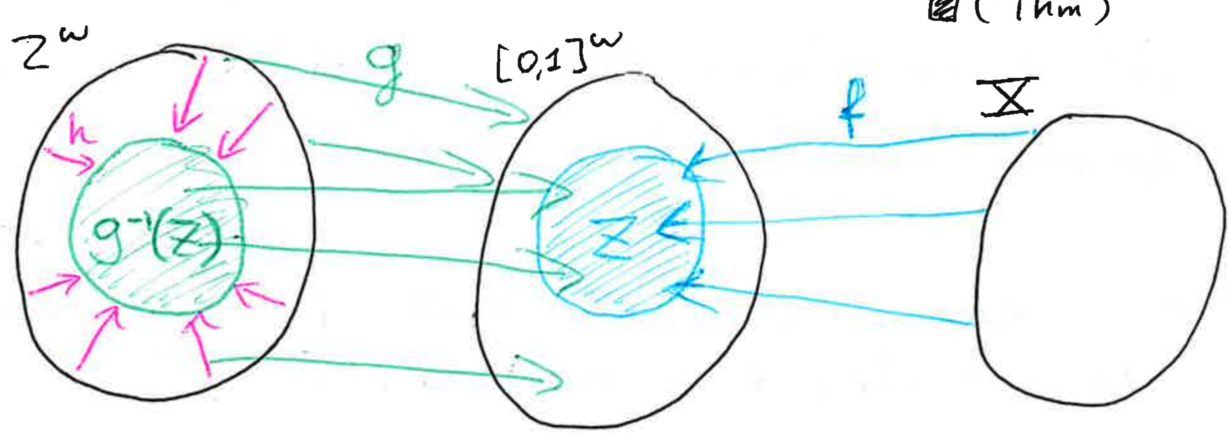
But $(2^\omega)^\omega \cong 2^\omega$. ∎(Prop)

Thm: Every non-$\emptyset$ compact metrizable space is a continuous image of 2^ω.

pf: Suppose that $X \neq \emptyset$ is compact metrizable. We proceed in three steps:

- Use earlier theorem to find a homeomorphism $f : X \cong Z$ with $Z \subseteq [0,1]^\omega$ compact.
- Fix a cont. surj $g : 2^\omega \to [0,1]^\omega$ from Prop.
- Since $g^{-1}(Z) \subseteq 2^\omega$ is closed and non-$\emptyset$, we can find a retraction $h : 2^\omega \to g^{-1}(Z)$.

Then the composition $f^{-1} \circ g \circ h : 2^\omega \to X$ works! ∎(Thm)



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Lecture 4

Polish subspaces of Polish spaces

We seek to analyze Polish spaces in analogy with our prior analysis of compact metrizable spaces.

<u>Warm-up A</u>: If $(X_n)_{n\in\omega}$ is a sequence of Polish spaces, then $\prod_n X$ (w/ fin supp prod top) is also Polish.

<u>pf (w-u A)</u>: <u>Separable</u>: ☺ ✓

<u>Completely metrizable</u>: Fix a summable sequence (ε_n) of positive reals, and for each $n\in\omega$ fix a compatible complete metric d_n on X_n bounded by 1.

Use $d : ((x_n), (y_n)) \longmapsto \sum_n \varepsilon_n d_n(x_n, y_n)$ ✓ ■ (w-u A)

<u>Warm-up B</u>: If X is Polish and $C \subseteq X$ is closed, then C is also Polish.

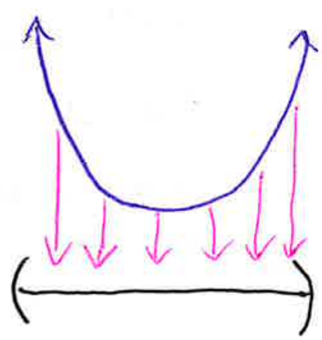
<u>pf (w-u B)</u>: <u>Separable</u>: Every subspace of X is separable. ✓ (using metrizability)

<u>Completely metrizable</u>: Just restrict any compatible complete metric on X. ✓ ■ (w-u B)

<u>Note</u>: The converse fails:

$(0,1) \subseteq [0,1]$

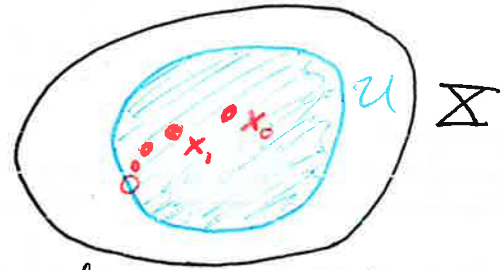
$\cong \mathbb{R}$.



closed subset of $\mathbb{R}^2$

② **Prop:** Suppose that $\mathbb{X}$ is Polish and $\mathcal{U} \subseteq \mathbb{X}$ is open. Then $\mathcal{U}$ is also Polish.

Concern: Some (Cauchy) seq (x_n) in $\mathcal{U}$ may converge to an element of $\mathbb{X} \setminus \mathcal{U}$.

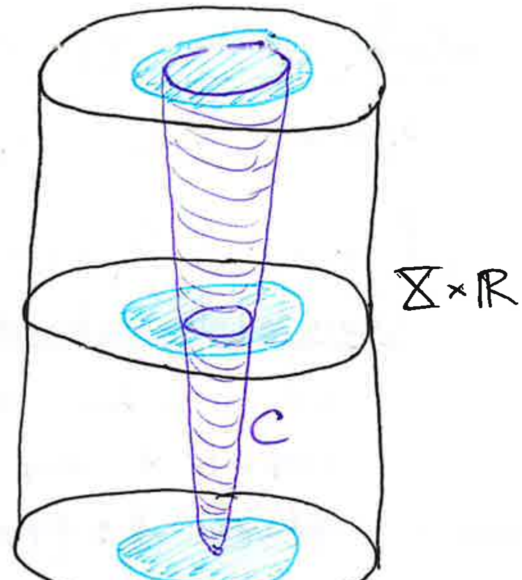


The only possible solution is to destroy Cauchyness.

pf (Prop): Fix a compatible complete metric d on $\mathbb{X}$. Put $F = \mathbb{X} \setminus \mathcal{U}$, so $\mathbb{X}$ is closed. Recall that

$$x \longmapsto d(x,F) = \inf\{d(x,y) : y \in F\}$$

is continuous, with $d(x,F) = 0$ iff $x \in F$. WLOG, $F \neq \emptyset$



Now, define $C \subseteq \mathbb{X} \times \mathbb{R}$ by

$$C = \left\{\left(x, \frac{1}{d(x,F)}\right) : x \in \mathcal{U}\right\}$$

By continuity of $x \longmapsto \frac{1}{d(x,F)}$, C is closed and hence Polish by W-u B.
Moreover, $C \cong \mathcal{U}$ via the projection map. ■ (Prop)

Digression: How to intersect like a $\boxed{\text{topologist.}}$

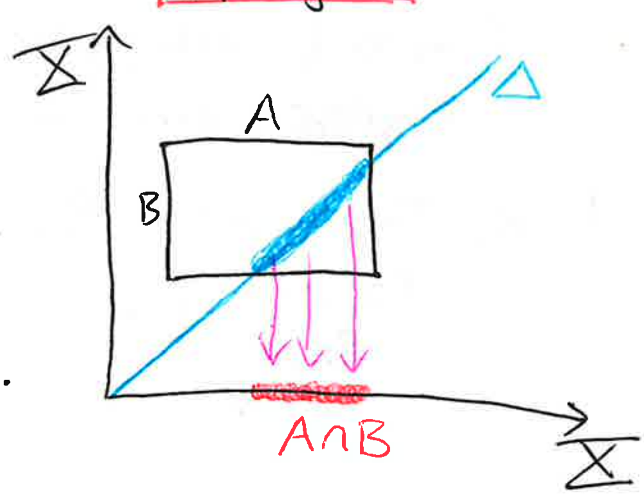
Suppose $A, B \subseteq \mathbb{X}$.

Then

$$A \cap B = \mathrm{Proj}\left[(A \times B) \cap \Delta\right]$$

where $\Delta = \{(x,x) : x \in \mathbb{X}\}$.

This works for big intersections.



③ **Prop:** Suppose that X is a Polish space and for each $n \in \omega$ we have Polish $A_n \subseteq X$. Then $\bigcap_n A_n$ is Polish.

pf: Via projection, we know $\bigcap_n A_n \cong (\prod_n A_n) \cap \Delta$. This is Polish by Warm-ups A & B. ∎(Prop)

Def: A subset $A \subseteq X$ is called G_δ if it is an intersection of a countable family of open sets.

Remark: In metrizable spaces, closed sets are G_δ.

Cor: If X is Polish and $A \subseteq X$ is G_δ, then A is Polish.

We shall soon see a converse. But first we discuss a convenient way of checking that a set is G_δ.

Def:
- Given a top space X, a metric space (Y, d), and a partial function $f: X \to Y$, its oscillation at $x \in X$ (possibly not in $\mathrm{dom}(f)$) is
$$\mathrm{osc}_f(x) = \inf\{\mathrm{diam}\, f[U] : x \in U \text{ and } U \text{ open}\}$$
- $x \in X$ is a continuity pt of f if $\mathrm{osc}_f(x) = 0$.

Remark: Equivalent metrics on Y yield the same continuity pts for a given $f: X \to Y$.

Prop: In the above setup, the set of continuity pts for any $f: X \to Y$ is G_δ in X.

pf: Put
$$A_\varepsilon = \{x \in X : \mathrm{osc}_f(x) < \varepsilon\}$$
$$= \bigcup \{U \subseteq X \text{ open} : \mathrm{diam}\, f[U] < \varepsilon\}$$
It's open! Hence (using some $\varepsilon_n \to 0$) $\bigcap_{\varepsilon > 0} A_\varepsilon$ is G_δ.

∎(Prop)

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Thm: X Polish and $A \subseteq X$. TFAE:

$\boxed{\mathrm{I}}$ A is Polish (with subspace topology)

$\boxed{\mathrm{II}}$ A is G_δ in X.

pf: $\boxed{\mathrm{II}} \Rightarrow \boxed{\mathrm{I}}$ ✓

$\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$: Fix a compatible complete metric d on A. Consider the partial function

$$f : X \to A, \quad a \mapsto a$$

Let G be its G_δ set of continuity points.

Claim: $A = \overline{A} \cap G$. where $\overline{A}$ denotes the closure of A

pf(C): It's not too hard to check $A \subseteq \overline{A} \cap G$, so we focus on the other containment. I.e., suppose $x \in \overline{A} \cap G$ with a goal of showing $x \in A$. Fix a sequence (x_n) from A with $\lim_n x_n = x$. Since $\mathrm{osc}_f(x) = 0$, for any $\varepsilon > 0$ we can find an open set $\mathcal{U}_\varepsilon$ about x with $\mathrm{diam}\, f[\mathcal{U}_\varepsilon] < \varepsilon$. But any such $f[\mathcal{U}_\varepsilon]$ contains a tail of (x_n). Hence, these tails have vanishing diameter, AKA (x_n) is Cauchy. Completeness of d then implies that $x = \lim_n x_n$ is in A as desired.

■ (C)

As both $\overline{A}$ and G are G_δ in X, it follows that $A = \overline{A} \cap G$ is also G_δ. ✓ ■ (Thm)

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Lecture 5

Intermission: Who cares about Polish spaces, anyway?

Analysts: Many objects of interest carry a natural Polish topology: L^p and ℓ^p spaces, separable Hilbert/Banach spaces, measures on compact metric space, ...

Probabilists: Brownian motion sample paths...

Algebraists: Many groups carry a Polish topology rendering the group operations continuous... Polish groups!

(a) $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \text{entries in } \mathbb{R}, \det 1\right\} \subseteq \mathbb{R}^4$ closed

(b) $S_\infty = \{\sigma : \omega \to \omega \text{ bijective}\} \subseteq \omega^\omega$ G_δ.

Logicians pretending to be algebraists:

You can encode all ctbly infinite groups within a Polish space: $2^{\omega\times\omega\times\omega}$. If Γ is a group with underlying set ω, associate a "code" $x_\Gamma \in 2^{\omega\times\omega\times\omega}$:

$x_\Gamma(\ell, m, n) = 1$ iff $\ell \cdot_\Gamma m = n$.

Problem: There are many codes for a group (up to isom.)

Solution: The group S_∞ acts on $2^{\omega\times\omega\times\omega}$:

$$(\sigma \cdot x)(\ell, m, n) = x(\sigma^{-1}(\ell), \sigma^{-1}(m), \sigma^{-1}(n))$$

Two codes are in the same orbit of this action exactly when they code isomorphic groups.

Upshot: Understanding ctbl groups up to isomorphism is tantamount to understanding (the orbits of) an action of a Polish group on a Polish space!

(2) OK, back to analyzing Polish spaces.
Let's glean some corollaries of our recent work.

<u>Cor</u>: Given a topological space X, TFAE:

$\boxed{\text{I}}$ X is Polish

$\boxed{\text{II}}$ X is homeomorphic to a G_δ subset of $[0,1]^\omega$.

$\boxed{\text{III}}$ X is homeomorphic to a G_δ subset of a compact metrizable space.

<u>Cor</u>: G_δ subsets of $\mathbb{R}$ cannot yield counterexamples to CH: any unctbl G_δ contains a Cantor set.

Our next goal:

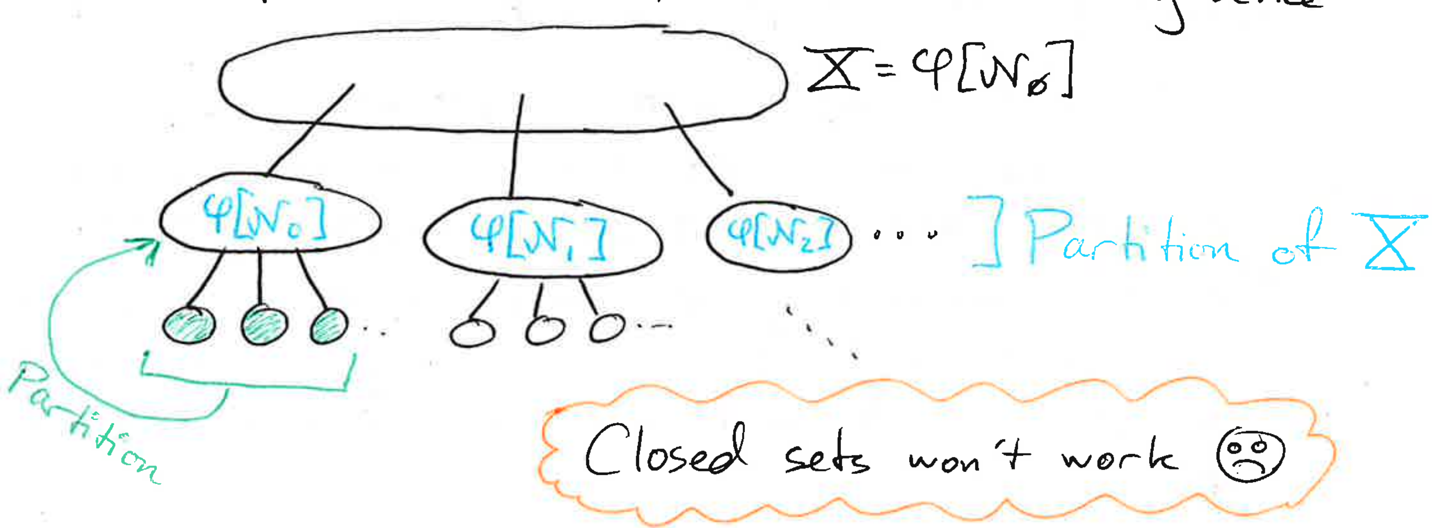
<u>Thm</u>: Suppose that X is Polish. Then there is a closed set $C \subseteq \omega^\omega$ and a continuous bijection

$$\varphi : C \to X.$$

<u>Cor</u>: Every non-$\emptyset$ Polish space is a continuous image of ω^ω.

<u>pf (Cor)</u>: Precompose φ from Thm with a retraction $\omega^\omega \to C$. ∎ (Cor)

<u>Idea</u>: To prove the Thm, we want something like:



③ **Def:** Given a topological space X, a subset $A \subseteq X$ is F_σ if it is a union of a ctbl family of closed sets. Equiv, if $X \setminus A$ is G_δ.

Splitting Lemma: Suppose that (X, d) is a separable metric space, that $F \subseteq X$ is F_σ, and that $\varepsilon > 0$. Then there is a sequence $(F_n)_{n \in \omega}$ of F_σ subsets of X such that:

ⓐ $F = \bigsqcup_n F_n$ ← $F = \bigcup_n F_n$ and the F_ns are pairwise disjoint.

ⓑ $\overline{F_n} \subseteq F$

ⓒ $\operatorname{diam}(\overline{F_n}) < \varepsilon$ when $F_n \neq \emptyset$.

pf (S.L.): Using separability, first find a sequence $(C_n)_{n \in \omega}$ of closed sets satisfying:

- $F = \bigcup_n C_n$
- $\operatorname{diam}(C_n) < \varepsilon$.

This is easy when F is closed; to handle F_σ just do ctbly many closed things and glue them into a big $(C_n)_{n \in \omega}$ sequence.

Next, disjointify: $F_n = C_n \setminus \bigcup_{m<n} C_m$.

Check that each F_n is indeed F_σ.

Now ⓐ $F = \bigcup_n C_n = \bigcup_n F_n$ ✓

ⓑ $\overline{F_n} \subseteq \overline{C_n} = C_n \subseteq F$ ✓

ⓒ $\operatorname{diam}(\overline{F_n}) \leq \operatorname{diam}(C_n) < \varepsilon$ ✓ ■ (S.L.)

(when non-$\emptyset$)

(4)

<u>pf (Thm):</u> Fix a compatible complete metric d on $\mathbb{X}$, and a positive sequence $\varepsilon_n \to 0$. Iterate the Splitting Lemma, recursively building for each $s \in \omega^{<\omega}$ an F_σ set $F_s \subseteq \mathbb{X}$ satisfying

- $F_\emptyset = \mathbb{X}$
- $F_s = \bigsqcup_n F_{s^\frown n}$
- $\overline{F_{s^\frown n}} \subseteq F_s$
- $\operatorname{diam}(F_s) < \varepsilon_{len(s)}$

Define a (pruned) tree $T \subseteq \omega^{<\omega}$ by

$$T = \{t \in \omega^{<\omega} : F_t \neq \emptyset\}$$

<u>Claim:</u> For each $\tau \in [T]$, $\bigcap_n F_{\tau \restriction n} = \bigcap_n \overline{F_{\tau \restriction n}}$.

<u>pf (C):</u> $\bigcap_n F_{\tau\restriction n} \subseteq \bigcap_n \overline{F_{\tau\restriction n}}$. ☺

On the other hand, $\bigcap_n \overline{F_{\tau\restriction n}} = \bigcap_n \overline{F_{\tau\restriction(n+1)}}$

$\subseteq \bigcap_n F_{\tau\restriction n}$. ∎(C)

So, for each $\tau \in [T]$, the set

$$\bigcap_n F_{\tau\restriction n}$$

is a singleton. Finally, put $C = [T]$ and define

$$\varphi : C \longrightarrow \mathbb{X}$$

so that $\bigcap_n F_{\tau\restriction n} = \{\varphi(\tau)\}$. ∎(Thm)

DST

Lecture 6

Baire category

Def: Given a top space X, we say that $A \subseteq X$ is nowhere dense if the only open subset of its closure $\overline{A}$ is $\emptyset$ [i.e., $\mathrm{int}(\overline{A}) = \emptyset$].
Equiv, if $X \setminus \overline{A}$ is dense.

Remark: Nowhere dense sets form an ideal: stable under finite unions and passage to subsets.

Def: $A \subseteq X$ is meager if it is a ctbl union of n.w. dense sets. Equiv, there are closed sets $C_n \subseteq X$ with $\mathrm{int}(C_n) = \emptyset$ and $A \subseteq \bigcup_n C_n$.

Remark: Meager sets form a σ-ideal: stable under ctbl unions and passage to subsets

(Polish) Baire Category Theorem:

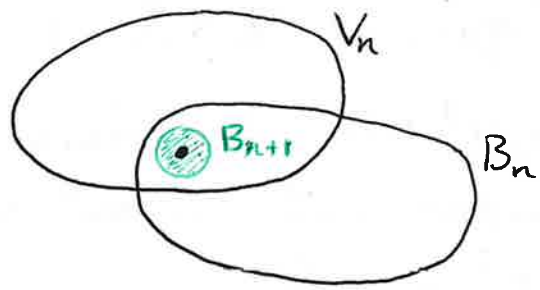
If X is Polish and $\mathcal{U} \subseteq X$ is non-$\emptyset$ open, then $\mathcal{U}$ is not meager.

Remark: Holds for Čech complete spaces, too!

(2) <u>pf (BCT)</u>: Fix a compatible complete metric d on X and positive $\varepsilon_n \to 0$. Given a sequence $(A_n)_{n \in \omega}$ of n.w. dense sets, we'll show $\mathcal{U} \nsubseteq \bigcup_n A_n$

Put $V_n = X \setminus \overline{A_n}$, which is open dense.
It suffices to show that $\mathcal{U} \cap \bigcap_n V_n \neq \emptyset$.
Recursively build non-$\emptyset$ open sets $B_n \subseteq \mathcal{U}$ s.t.

- $B_0 = \mathcal{U}$
- $B_{n+1} \subseteq B_n \cap V_n$
- $\overline{B_{n+1}} \subseteq B_n$
- $\operatorname{diam}(\overline{B_{n+1}}) < \varepsilon_n$



Then $\bigcap_n B_n = \bigcap_n \overline{B_n}$ is non-$\emptyset$.
But $\bigcap_n B_n \subseteq \mathcal{U} \cap \bigcap_n V_n$, and we are done. ∎(BCT)

<u>Def</u>: $A \subseteq X$ is <u>comeager</u> if $X \setminus A$ is meager.
DeMorganizing, this is equivalent to containing an intersection of ctbly many dense open sets.

<u>Remark</u>: In Polish spaces, BCT implies that a set is comeager iff it contains a dense G_δ.

<u>Def</u>: Given a top space X, we say that $A \subseteq X$ has the <u>property of Baire</u> (BP) if there is open $\mathcal{U} \subseteq X$ such that $A \triangle \mathcal{U}$ is meager.
"A is a meager set away from being open."

③ Fun facts about BP sets (with justification?!)

ⓐ Open sets are BP. ☺

ⓑ Closed sets are also BP: If C is closed, then $C \triangle \mathrm{int}(C) = C \setminus \mathrm{int}(C)$ is a closed set with empty interior, hence $C \triangle \mathrm{int}(C)$ is meager.

ⓒ If $A \triangle B$ is meager and B is BP, then A is also BP.

ⓓ If A is BP, then $X \setminus A$ is also BP

Fix an open set $\mathcal{U}$ with $A \triangle \mathcal{U}$ meager. Then

$$(X \setminus A) \triangle \underbrace{(X \setminus \mathcal{U})}_{\text{closed?}} = A \triangle \mathcal{U} \text{ is (still) meager.}$$

Now ⓑ + ⓒ imply that $X \setminus A$ is BP.

ⓔ If $(A_n)_{n \in \omega}$ are all BP, so is $\bigcup_n A_n$:

Fix open $\mathcal{U}_n$ with $A_n \triangle \mathcal{U}_n$ meager.

Then $(\bigcup_n A_n) \triangle (\bigcup_n \mathcal{U}_n) \subseteq \bigcup_n (A_n \triangle \mathcal{U}_n)$ is meager.

Def: We say that $\mathcal{A} \subseteq \mathcal{P}(X)$ is a <u>σ-algebra</u> if it is a non-$\emptyset$ collection stable under:

- complementation
- countable union
- countable intersection

redundant

④ So our fun facts have proved:

Thm: In any topological space, the BP sets form a σ-algebra containing every open set.

Def: Given a top space X, the Borel σ-algebra is the smallest σ-algebra containing every open set. We colloquially refer to its elements as Borel (sub)sets of X.

Cor: In any top space, Borel sets are BP.

Remark: In fact, BP is the join of Borel and meager.

Localization: If X is a top space with base $\mathcal{B}$ and $A \subseteq X$ is BP and nonmeager, then there is some non-$\emptyset$ $\mathcal{U} \in \mathcal{B}$ in which A is comeager. I.e., such that $\mathcal{U} \setminus A$ is meager.

pf: Fix open V with $A \triangle V$ meager. Nonmeagerness of A implies $V \neq \emptyset$. Now pick any non-$\emptyset$ $\mathcal{U} \in \mathcal{B}$ with $\mathcal{U} \subseteq V$. ∎ (Localization)

Remark: Localization is analogous with "Lebesgue density" in say $\mathbb{R}$: given $A \subseteq \mathbb{R}$ with $\mu(A) > 0$ and $\varepsilon > 0$, there is a non-$\emptyset$ open interval $I \subseteq \mathbb{R}$ with

$$\frac{\mu(A \cap I)}{\mu(I)} > 1 - \varepsilon.$$

DST

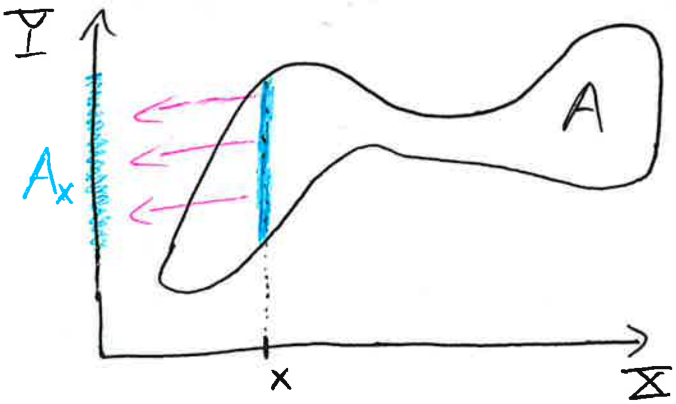
Lecture 7

Def: Given $A \subseteq X \times Y$ and $x \in X$, we define the <u>x^{th} vertical section</u> of A as the set

$$A_x = \{y \in Y : (x, y) \in A\}.$$

Analogously, for $y \in Y$ the <u>y^{th} horizontal section</u> is

$$A^y = \{x \in X : (x, y) \in A\}.$$



Def: $\forall^* x \in X$ blah(x) abbrevs "$\{x \in X : \text{blah}(x)\}$ is comeager in X."
$\exists^* x \in X$ blah(x) abbrevs "$\{x \in X : \text{blah}(x)\}$ is nonmeager in X."

Thm (Kuratowski-Ulam): Suppose that X, Y are top spaces and that Y is 2^{nd} ctbl. I.e, Y has a ctbl base
Suppose further that $A \subseteq X \times Y$ is BP. TFAE:

$\boxed{\text{I}}$ A is comeager in $X \times Y$.

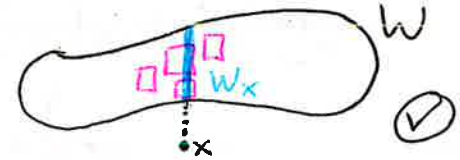
$\boxed{\text{II}}$ $\forall^* x \in X$ A_x is comeager in Y.

Remark: For convenience, I will assume X and Y satisfy BCT: non-$\emptyset$ open sets are nonmeager. This assumption is unnecessary (HW?).

(2) pf (K-U, BCT version): Enumerate a ctbl base $\{V_n : n \in \omega\}$ for the top on Y. WLOG $V_n \neq \emptyset$.

Lemma: If $W \subseteq X \times Y$ is open dense, then $\forall^* x \in X$ W_x is open dense in Y.

pf($\mathcal{L}$): First, $\forall x \in X$ W_x is open:



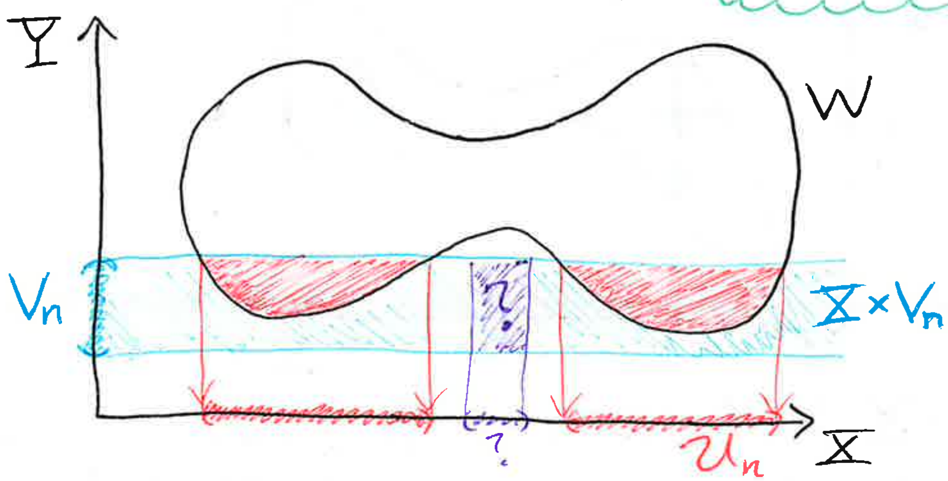
✓

So we just need to check density.

Note that $B \subseteq Y$ is dense iff $\forall n \in \omega$ $B \cap V_n \neq \emptyset$.

Put $\mathcal{U}_n = \mathrm{Proj}_X [W \cap (X \times V_n)]$

$x \in \mathcal{U}_n$ iff $W_x \cap V_n \neq \emptyset$



As illustrated, $\mathcal{U}_n$ is dense since W meets every open rectangle in $X \times Y$. It is also open as projections map open sets to open sets.

Finally, $x \in \bigcap_n \mathcal{U}_n \Rightarrow W_x$ is dense, so $\forall^* x \in X$ W_x is open dense, as desired. ∎($\mathcal{L}$)

③ <u>pf</u> (K-U, BCT version, cont.)

$\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$: Since A is comeager, we may fix open dense $W_m \subseteq X \times Y$ with $\bigcap_m W_m \subseteq A$. By the Lemma, for each m there is comeager $C_m \subseteq X$ such that $x \in C_m \Rightarrow (W_m)_x$ is comeager. Intersecting all these, $x \in \bigcap_m C_m$ implies that $\bigcap_m (W_m)_x$ is comeager. But $\bigcap_m (W_m)_x \subseteq A_x$ and hence $\forall^* x \in X$ A_x is comeager. ✓

$\neg\boxed{\mathrm{I}} \Rightarrow \neg\boxed{\mathrm{II}}$: Suppose that $A \subseteq X \times Y$ is BP and NOT comeager. Then $B = (X \times Y) \setminus A$ is BP and nonmeager. Localize, finding non-$\emptyset$ open $U \subseteq X$ and $V \subseteq Y$ s.t. B is comeager in $U \times V$. We now apply the $\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$ argument to $B \cap (U \times V)$ as a comeager subset of $U \times V$:

$$\forall^* x \in U \quad B_x \text{ is comeager in } V, \text{ i.e.,}$$
$$\forall^* x \in U \quad A_x \text{ is meager in } V.$$

As $\boxed{U \text{ and } V \text{ are nonmeager}}$ (BCT), this shows that a nonmeager set of x FAIL to have comeager A_x, establishing $\neg\boxed{\mathrm{II}}$. ✓ ∎ (K-U, BCT version)

<u>Remark</u>: To avoid assuming BCT, we need to localize a bit more carefully to ensure that our $U \subseteq X$ and $V \subseteq Y$ are nonmeager.

(4) Fun with quantifiers

(a) Suppose that $\mathbb{X}$ is a 2^{nd} ctbl top space, and enumerate the nonmeager basic open sets as $\{\mathcal{U}_n : n \in \omega\}$.

We may think of $A \subseteq \mathbb{X}$ as a "unary predicate"

$$A(x) \text{ iff } x \in A.$$

For BP $A \subseteq \mathbb{X}$, we may regard localization as a manipulation of quantifiers:

- $\exists^* x \in \mathbb{X}\ A(x)$ iff $\exists n \in \omega\ \forall^* x \in \mathcal{U}_n\ A(x)$
- $\forall^* x \in \mathbb{X}\ A(x)$ iff $\forall n \in \omega\ \exists^* x \in \mathcal{U}_n\ A(x)$.

(b) Similarly, if $\mathbb{X}$ and $\mathbb{Y}$ are both 2^{nd} ctbl top spaces, we may view Kuratowski-Ulam as quantifier manipulation. For BP $A \subseteq \mathbb{X} \times \mathbb{Y}$:

$$\forall^* x \in \mathbb{X}\ \forall^* y \in \mathbb{Y}\ A(x,y)$$

iff

$$\forall^* (x,y) \in \mathbb{X} \times \mathbb{Y}\ A(x,y)$$

iff

$$\forall^* y \in \mathbb{Y}\ \forall^* x \in \mathbb{X}\ A(x,y).$$

(1)

DST

Lecture 8

Applications of Baire category on 2^ω

Def: The equivalence relation $\mathbb{E}_0$ on 2^ω is defined

$$x \,\mathbb{E}_0\, y \text{ iff } \exists m \ \forall n \geq m \ \ x(n) = y(n).$$

"x and y eventually agree" or "agree mod finite"

$$\begin{array}{l} \\ x = \underline{?}\ \underline{?}\ \underline{?}\ \underline{?} \\ y = \underline{?}\ \underline{?}\ \underline{?}\ \underline{?} \end{array} \begin{array}{l} m \\ \boxed{\begin{array}{l} \underline{0}\ \underline{0}\ \underline{1}\ \underline{0} \ \cdots \\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{0} \ \cdots \end{array}} \end{array}$$

Note: If F_m on 2^ω is defined by

$$x \,F_m\, y \text{ iff } \forall n \geq m \ \ x(n) = y(n)$$

then F_m is a closed subset of $2^\omega \times 2^\omega$.

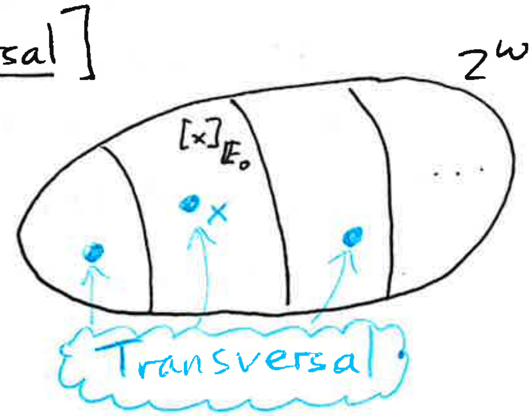
Since $\mathbb{E}_0 = \bigcup_m F_m$, $\mathbb{E}_0$ is F_σ [but not G_δ. HW?]

Another perspective: $\mathbb{E}_0$ is the orbit equivalence relation of the action $(\mathbb{Z}/2\mathbb{Z})^{<\omega} \curvearrowright 2^\omega$ by homeomorphisms.

Prop: There is no BP subset of 2^ω that meets each $\mathbb{E}_0$-class in exactly one point.

[I.e., $\mathbb{E}_0$ admits no BP transversal]

"Vitali for Baire category."



② **pf (Prop):** Towards a contradiction, suppose that $A \subseteq 2^\omega$ is a BP transversal. It's either meager or it isn't.

__Case 0:__ A is meager. Then for each $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{<\omega}$ we know $\gamma \cdot A$ is also meager as homeos preserve Baire category. But then we have $2^\omega = \bigcup_\gamma \gamma \cdot A$ is meager, contradicting BCT. ✓

__Case 1:__ A is nonmeager. Localize, finding $s \in 2^{<\omega}$ so that A is comeager in $\mathcal{N}_s$. Fix $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{<\omega}$ that flips a bit AFTER s (and nothing else):

Note that $\gamma \cdot \mathcal{N}_s = \mathcal{N}_s$. As γ is a homeo, $\gamma \cdot A$ is comeager in $\gamma \cdot \mathcal{N}_s = \mathcal{N}_s$. In particular, $A \cap \gamma \cdot A \neq \emptyset$ as both are comeager in $\mathcal{N}_s$ (BCT). Fix $x \in A \cap \gamma \cdot A$, so that x and $\gamma \cdot x$ are both in A. This contradicts A being a transversal. ✓ ■(Prop)

__Remarks:__

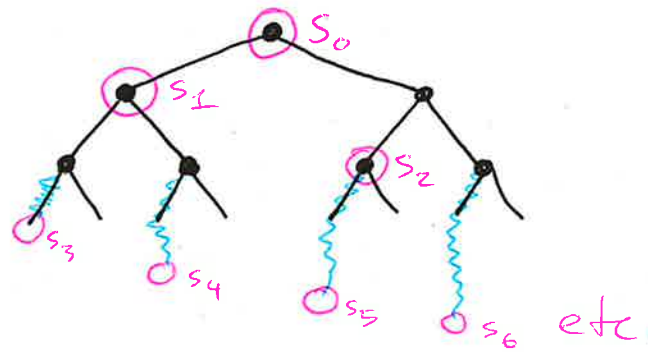
ⓐ Shelah (1984, following Solovay) showed that if ZF is consistent, so is ZF + DC + "all subsets of 2^ω are BP."
This means that ZF + DC is insufficient to build a transversal of $\mathbb{E}_0$ (or math is broken)

ⓑ We will later see that $\mathbb{E}_0$ is the __canonical impediment__ to finding definable transversals.

③ Now for some graph theory...

We (following Kechris-Solecki-Todorcevic, 1999) will define a special graph $\mathbb{G}_0$ with vertex set 2^ω.

Fix for each $n \in \omega$ a string $s_n \in 2^n$ so that $\{s_n : n \in \omega\}$ is <u>dense</u>: $\forall t \in 2^{<\omega}\ \exists n \in \omega\ t \subseteq s_n$.



<u>Def:</u> $\mathbb{G}_0$ is "the" graph on 2^ω with an edge between x and y iff $\exists n \in \omega\ \exists z \in 2^\omega$ s.t.

$$x = s_n {}^\frown 0 {}^\frown z$$
$$y = s_n {}^\frown 1 {}^\frown z$$

Equiv, x and y disagree in exactly one bit, and the common prefix before this is an s_n.

<u>Remark:</u> $\mathbb{G}_0$ is F_σ, acyclic, and its connectedness relation is $\mathbb{E}_0$.

<u>Def:</u> Given a graph G on X, we say that $A \subseteq X$ is <u>G-independent</u> if there is no G-edge between elements of A. Viewing $G \subseteq X^2$, this means

$$G \cap (A \times A) = \emptyset.$$

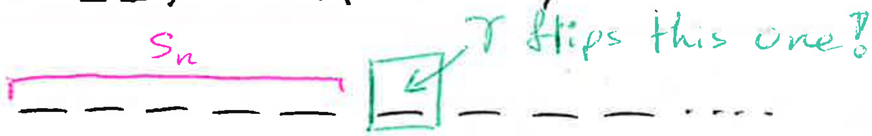
<u>Prop:</u> If $A \subseteq 2^\omega$ is BP and $\mathbb{G}_0$-independent, then A is meager.

<u>Remark:</u> The previous Prop may be restated as:

"If $A \subseteq 2^\omega$ is BP and $(\mathbb{E}_0 \setminus \Delta)$-indep, then A is meager."

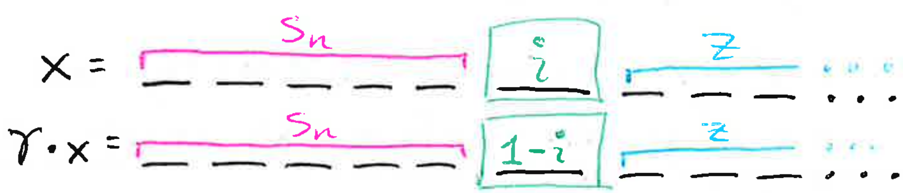
④ pf (Prop): Contrapose: Suppose that $A \subseteq 2^\omega$ is BP nonmeager, aiming to find a $\mathbb{G}_0$-edge between elements of A. Localize, finding $t \in 2^{<\omega}$ so that A is comeager in $\mathcal{N}_t$. Find $n \in \omega$ with $t \sqsubseteq s_n$. So $\mathcal{N}_{s_n} \subseteq \mathcal{N}_t$ and thus A is again comeager in $\mathcal{N}_{s_n}$.

Let $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{<\omega}$ flip exactly bit n:



So $\gamma \cdot A$ is still comeager in $\gamma \cdot \mathcal{N}_{s_n} = \mathcal{N}_{s_n}$.

As before, $A \cap \gamma \cdot A \neq \emptyset$ by BCT. Fix $x \in A \cap \gamma \cdot A$.



This is a $\mathbb{G}_0$-edge! ■ (Prop)

<u>Remarks:</u>

ⓐ $\mathbb{G}_0$ is acyclic, so ZFC proves it is 2-colorable. But a BP coloring requires uncountably many colors.

ⓑ We will soon see that $\mathbb{G}_0$ is a <u>canonical impediment</u> to definably coloring with countably many colors.

ⓒ If μ is the $(1/2, 1/2)$-product measure on 2^ω, ZFC proves that $\mathbb{G}_0$ is μ-measurably $\boxed{\text{3-colorable.}}$

①

DST

Lecture 9

This week, we contrast Borel vs analytic sets in Hausdorff $\mathbb{X}$.

Def: A set is Borel if it is in the σ-algebra generated by the topology on $\mathbb{X}$.

Def: A set is analytic if it is empty or the image of a continuous function $\omega^\omega \to \mathbb{X}$.

What does a (non-$\emptyset$) analytic set "look like?"

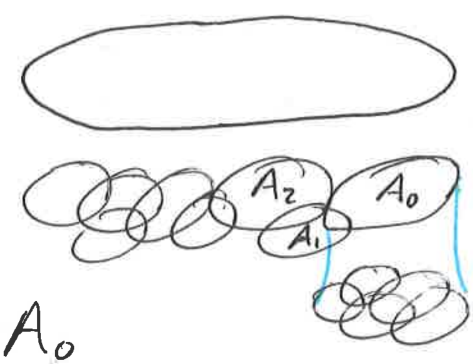
Fix continuous $\varphi : \omega^\omega \to \mathbb{X}$ with $\varphi[\omega^\omega] = A$.

The sets $A_s = \varphi[N_s]$, $s \in \omega^{<\omega}$, give A bonus structure:

$A_\emptyset = A$

$\bigcup_n A_n = A$

$\bigcup_n A_{0^\frown n} = A_0$

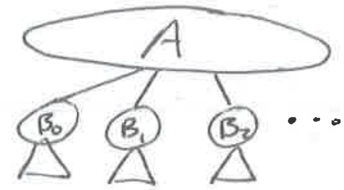


Note: at each level, the family $\{A_s : s \in \omega^n\}$ need not be p.w. disj, as φ is not assumed to be injective!

- In general, $A_s = \bigcup_n A_{s^\frown n}$.
- Given a metric on $\mathbb{X}$, continuity ensures for $\sigma \in \omega^\omega$ that $\mathrm{diam}(A_{\sigma \upharpoonright n}) \to 0$.
- More generally, continuity says that for any open nbhd $\mathcal{U}$ about $\varphi(\sigma)$, for suff large n we have $A_{\sigma\upharpoonright n} \subseteq \mathcal{U}$.
- In particular, $\bigcap_n A_{\sigma \upharpoonright n} = \bigcap_n \overline{A_{\sigma\upharpoonright n}} = \{\varphi(\sigma)\}$.

② **Prop**: The union of countably many analytic sets is again analytic.

pf: Suppose $(B_n)_{n\in\omega}$ are all analytic, and $A = \bigcup_n B_n$

Convert schemes  into scheme 

WLOG, all $B_n \neq \emptyset$. Fix continuous surjections $\varphi_n : \omega^\omega \twoheadrightarrow B_n$. Amalgamate into continuous function

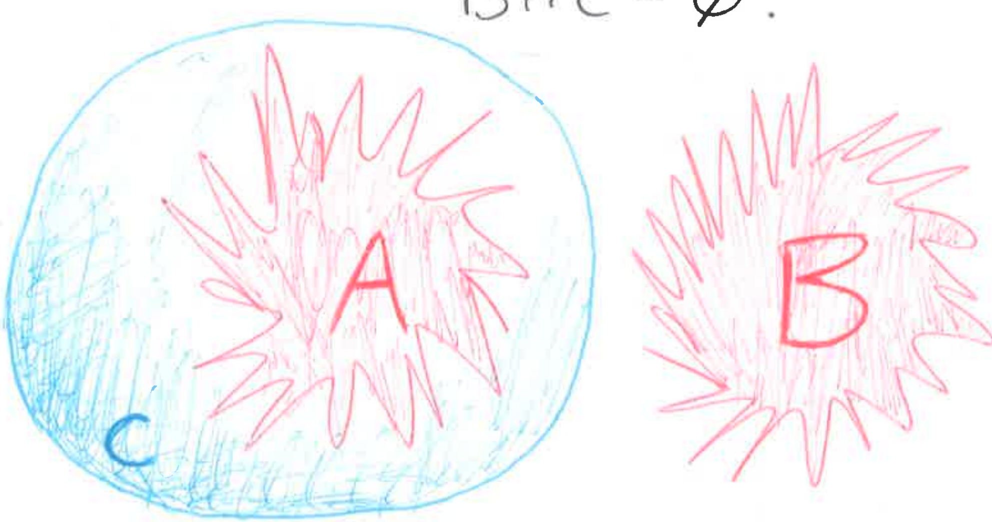
$$\Psi : \omega^\omega \longrightarrow \mathbb{X} \qquad \text{Check } \Psi[\omega^\omega] = A.$$
$$n^\frown \sigma \longmapsto \varphi_n(\sigma).$$

■ (Prop)

Thm (Lusin): Any two disjoint analytic subsets of $\mathbb{X}$ may be separated by a Borel set.

That is, given analytic sets $A, B \subseteq \mathbb{X}$ with $A \cap B = \emptyset$, there is a Borel set $C \subseteq \mathbb{X}$ s.t.

$$A \subseteq C \text{ and}$$
$$B \cap C = \emptyset.$$



③ <u>pf</u> (Thm): WLOG, assume $A, B \neq \emptyset$. Fix continuous $\varphi_A, \varphi_B : \omega^\omega \to X$ with $\varphi_A[\omega^\omega] = A$
$\varphi_B[\omega^\omega] = B$.

Consider corresponding schemes $A_s = \varphi_A[N_s]$
$s \in \omega^{<\omega}$ $B_s = \varphi_B[N_s]$.

All are analytic, $A_s = \bigcup_n A_{s^\frown n}$, etc.
Let's say that a pair $(s,t) \in (\omega^{<\omega})^2$ is GOOD if A_s and B_t can be separated by a Borel set; else, call (s,t) BAD. We hope $(\emptyset, \emptyset)$ is GOOD.

<u>Lemma</u>: If (s,t) is BAD, then there are $m, n \in \omega$ s.t. $(s^\frown m, t^\frown n)$ remains BAD.

<u>pf ($\mathcal{L}$)</u>: Twds a contr, suppose all such pairs are GOOD. Fix Borel $C_{m,n}$ s.t. $A_{s^\frown m} \subseteq C_{m,n}$ & $B_{t^\frown n} \cap C_{m,n} = \emptyset$. Check that $\bigcup_m \bigcap_n C_{m,n}$ separates A_s and B_t, $\Rightarrow\!\Leftarrow$ ■($\mathcal{L}$)

Twds a contr, suppose that $(\emptyset, \emptyset)$ is BAD. Iterate Lemma to build $\sigma, \tau \in \omega^\omega$ s.t. $\forall n$. $(\sigma \restriction n, \tau \restriction n)$ is BAD. Certainly $\varphi_A(\sigma) \in A$ and $\varphi_B(\tau) \in B$, hence $\varphi_A(\sigma) \neq \varphi_B(\tau)$. Hausdorffness yields disjoint open $\mathcal{U}, V$ about $\varphi_A(\sigma), \varphi_B(\tau)$. For large n,

$A_{\sigma \restriction n} \subseteq \mathcal{U}$
$B_{\tau \restriction n} \subseteq V$

contradicting BADness of $(\sigma \restriction n, \tau \restriction n)$. ■(Thm)

Cor (Suslin): If A and $X \setminus A$ are both analytic, then A is Borel.

pf: Separate A from $X \setminus A$ (only A works). $\blacksquare$ (Cor)

Remark: Analytic sets are sometimes called ω-Suslin

Def: $A \subseteq X$ is ω-Lusin if there is closed $C \subseteq \omega^\omega$ and cont $\boxed{\text{injection}}$ $\varphi : C \to X$ with $\varphi[C] = A$.

Recall: Polish spaces are ω-Lusin (in retrospect).

Thm (Lusin-Suslin): Every ω-Lusin set is Borel.

pf: Fix $A \subseteq X$, $C \subseteq \omega^\omega$ closed, $\varphi : C \to X$ cont. inj. with $\varphi[C] = A$. Fix also a pruned tree $T \subseteq \omega^{<\omega}$ with $C = [T]$. Define as usual a scheme $A_s = \varphi[N_s]$, so $A_s \neq \emptyset$ iff $s \in T$. Each A_s is analytic.

Injectivity of φ ensures for $s \neq t \in \omega^n$ that $A_s \cap A_t = \emptyset$. Separating all of these pairs yields Borel $B_s \subseteq X$ s.t.:

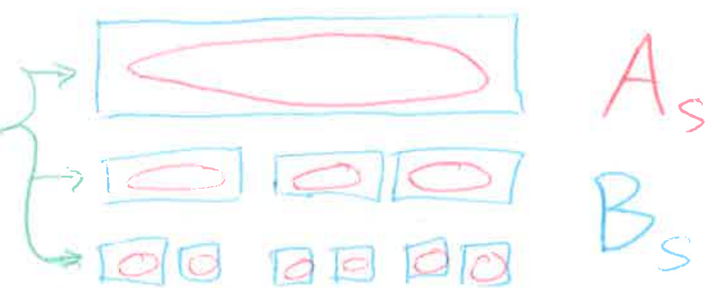
- $A_s \subseteq B_s \subseteq \overline{A_s}$ — Replace B_s by $B_s \cap \overline{A_s}$
- $s \neq t \in \omega^n \Rightarrow B_s \cap B_t = \emptyset$
- $B_{s^\frown n} \subseteq B_s$ — Replace $B_{s^\frown n}$ by $B_{s^\frown n} \cap B_s$

Put $B = \bigcap\limits_{n \in \omega} \bigcup\limits_{s \in \omega^n} B_s$, which is Borel.

union each level

intersect across levels

Next time: $\boxed{A = B}$



~To be continued~

DST

Lecture 10

pf (Thm, cont.): We had ω-Lusin A and Borel $B = \bigcap_{n\in\omega} \bigcup_{s\in\omega^n} B_s$

Claim: $A = B$.

pf(C): $A \subseteq B$: Given $\sigma \in [T]$, we want $\varphi(\sigma) \in B$.
Check $\forall n \in \omega$ $\varphi(\sigma) \in A_{\sigma\restriction n} \subseteq B_{\sigma\restriction n} \subseteq \bigcup_{s\in\omega^n} B_s$. ✓

$B \subseteq A$: Fix $x \in B$. $\forall n \in \omega$ $\exists! s_n \in \omega^n$ with $x \in B_{s_n}$.
Moreover, $s_n \sqsubseteq s_{n+1}$. So $\sigma = \bigcup_n s_n$ is a branch in T. Finally, $x \in \bigcap_n \overline{A_{\sigma\restriction n}} = \{\varphi(\sigma)\}$, and thus $x \in A$. ✓ ∎(C)

∎(Thm)

Recap: Suppose that X is Hausdorff.

- If A and $X \setminus A$ are analytic, then A is Borel.
- More generally, disjoint analytic sets may be separated by a Borel set.
- ω-Lusin subsets of X are Borel.
- Polish spaces are ω-Lusin.

Remark: Replacing ω^ω by κ^ω everywhere yields κ-Suslin and κ-Lusin sets...

Today's goal: Understanding ω-Lusin subsets of Polish spaces.

② But first...

Lemma: Suppose that X is a topological space, and that $\mathcal{A} \subseteq \mathcal{P}(X)$ satisfies:

- every open set is in $\mathcal{A}$
- every closed set is in $\mathcal{A}$
- $\mathcal{A}$ is stable under ctbl intersections
- $\mathcal{A}$ is stable under ctbl disjoint unions

Then every Borel set is in $\mathcal{A}$.

pf($\mathcal{L}$): Define $\mathcal{B} \subseteq \mathcal{A}$ by

$$B \in \mathcal{B} \text{ iff } (B \in \mathcal{A} \text{ and } X \setminus B \in \mathcal{A}).$$

Let's show that $\mathcal{B}$ is a σ-algebra containing all open sets. Then $\mathcal{B}$ (and hence $\mathcal{A}$) will contain all Borel sets, too, granting the lemma.

ⓐ Open sets are in $\mathcal{B}$. ✓

ⓑ $\mathcal{B}$ is stable under complementation. ✓

ⓒ Countable unions: Fix $B_n \in \mathcal{B}$ and put $B = \bigcup_n B_n$.
We want $B \in \mathcal{B}$, i.e., $B \in \mathcal{A}$ and $X \setminus B \in \mathcal{A}$.

$B \in \mathcal{A}$: Put $C_n = B_n \cap \bigcap_{i<n} (X \setminus B_i)$, so each $C_n \in \mathcal{A}$. Now $B = \bigsqcup_n C_n$, so $B \in \mathcal{A}$. ✓

$X \setminus B \in \mathcal{A}$: $X \setminus B = \bigcap_n (X \setminus B_n) \in \mathcal{A}$ ✓ ✓

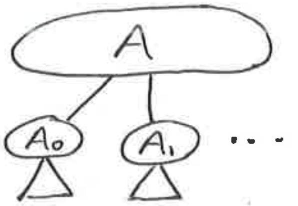
We did it! $\blacksquare(\mathcal{L})$

③

Thm: Suppose that X is Polish and $A \subseteq X$ is Borel. Then A is ω-Lusin.

pf: We check that our lemma's criteria hold of
$$\mathcal{A} = \{A \subseteq X : A \text{ is } \omega\text{-Lusin}\}.$$

- Open/closed sets are in $\mathcal{A}$:
  We know that G_δ subsets are Polish, thus ω-Lusin. ✓
- $\mathcal{A}$ stable under ctbl intersections:
  Use our projection trick: $\bigcap_n A_n = \mathrm{Proj}[(\prod_n A_n) \cap \Delta]$.
  Fix ω-Lusin $A_n \subseteq X$ witnessed by closed $C_n \subseteq \omega^\omega$ and cont. bijection $\varphi_n : C_n \to A_n$. Induce
  $$\prod_n \varphi_n : \prod_n C_n \longrightarrow \prod_n A_n \subseteq X^\omega.$$
  Then $C = (\prod_n \varphi_n)^{-1}(\Delta) \subseteq (\omega^\omega)^\omega$ is closed.
  Finally, $\mathrm{Proj} \circ \prod_n \varphi_n : C \to X$ is a cont. injection with image $\bigcap_n A_n$ as desired. ✓
- $\mathcal{A}$ stable under ctbl disjoint unions:



Suppose $(A_n)_{n\in\omega}$ is a p.w. disjoint sequence of ω-Lusin subsets of X. Fix closed sets $C_n \subseteq \omega^\omega$ and cont. bijections $\varphi_n : C_n \to A_n$. Now put $C = \{n^\frown\sigma : \sigma \in C_n\} \subseteq \omega^\omega$, which is closed, and check that
$$\begin{aligned} \varphi : C &\to X \\ n^\frown\sigma &\mapsto \varphi_n(\sigma) \end{aligned}$$
witnesses that A is ω-Lusin. ✓

We did it again! ☒(Thm)

4 So subsets of Polish spaces are ω-Lusin iff they are Borel. This is very powerful.

Cor: Suppose that X and Y are Polish and that $f : X \to Y$ is continuous and injective. Then for any Borel $A \subseteq X$, $f[A]$ is also Borel.

pf: If φ witnesses ω-Lusinity of A, then $f \circ \varphi$ witnesses ω-Lusinity of $f[A]$. ∎(Cor)

Cor (Alexandrov & Hausdorff): Suppo e that X is Polish and that $A \subseteq X$ is Borel. Exactly one:

$\boxed{\text{I}}$ A is countable

$\boxed{\text{II}}$ There is a continuous injection $2^\omega \hookrightarrow A$.

pf (Cor): Suppose that A is uncountable. Witness its ω-Lusinity with closed $C \subseteq \omega^\omega$ and a continuous bijection $\varphi : C \to A$.

Note that C is also unctbl, so Cantor grants a cont. inj. $f : 2^\omega \hookrightarrow C$. Then

$$\varphi \circ f : 2^\omega \longrightarrow A$$

is a cont. inj. as in $\boxed{\text{II}}$. ∎(Cor)

DST

Lecture 11

Recap: X Polish and $A \subseteq X$. TFAE:

$\boxed{\text{I}}$ A is Borel

$\boxed{\text{II}}$ A is ω-Lusin

$\boxed{\text{III}}$ A and $X \setminus A$ are both analytic [ω-Suslin]

Changing topology

First, some impossible-to-remember terminology:
Given two topologies $\tau_0 \subseteq \tau_1$ on some set X, we say τ_0 coarsens τ_1, or τ_1 refines τ_0.

Def: Suppose that C is a top space, X is a set, and $\varphi : C \to X$. The push-forward topology τ_φ on X is defined by

$$\mathcal{U} \in \tau_\varphi \text{ iff } \varphi^{-1}(\mathcal{U}) \text{ is open in } C.$$

as always, $\varphi^{-1}(\mathcal{U}) = \{c \in C : \varphi(c) \in \mathcal{U}\}$.

Remark: In other words, τ_φ is the finest topology on X rendering the map φ continuous.

Def: Given a topology τ on X, we shall denote by $\mathcal{B}(\tau)$ its corresponding Borel σ-algebra.

② **Prop:** Suppose that $C \subseteq \omega^\omega$ is closed, that X is a set, and that $\varphi : C \to X$ is a function.

ⓐ If φ is bijective, then (X, τ_φ) is Polish.

ⓑ If τ is a top on X s.t. φ is continuous, then $\tau \subseteq \tau_\varphi$.

ⓒ If φ is bijective and τ is a Polish top on X s.t. φ is cont., then $\mathcal{B}(\tau_\varphi) = \mathcal{B}(\tau)$.

pf: ⓐ φ is a homeomorphism. ✓

ⓑ See previous remark. ✓

ⓒ Since $\tau_\varphi \supseteq \tau$, we know $\mathcal{B}(\tau_\varphi) \supseteq \mathcal{B}(\tau)$. Towards checking $\mathcal{B}(\tau_\varphi) \subseteq \mathcal{B}(\tau)$, suppose that $B \in \mathcal{B}(\tau_\varphi)$. This means that $A = \varphi^{-1}(B)$ is a Borel subset of C. But then $B = \varphi[A]$ is a continuous injective image of a Borel set. We conclude $B \in \mathcal{B}(\tau)$. ✓

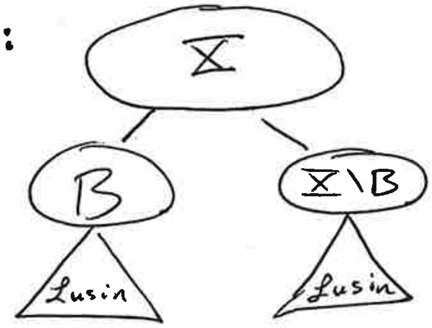
■ (Prop)

Thm: Suppose that (X, τ) is Polish and $B \in \mathcal{B}(\tau)$. Then there is a finer Polish topology $\tau_B \supseteq \tau$ on X s.t.:

- $\mathcal{B}(\tau_B) = \mathcal{B}(\tau)$
- B is τ_B-clopen.

Remark: Next time we will see that $\tau_0 \subseteq \tau$ automatically implies $\mathcal{B}(\tau_0) = \mathcal{B}(\tau_1)$ when both topologies are Polish!

③ pf (Thm):



More formally, fix closed $C_0, C_1 \subseteq \omega^\omega$ and cont. injections $\varphi_i : C_i \to X$ with
$$\varphi_0[C_0] = B$$
$$\varphi_1[C_1] = X \setminus B.$$

Put $D = \{ i^\frown \sigma : \sigma \in C_i \} \subseteq \omega^\omega$ closed, and
$$\Psi : D \to X$$
$$i^\frown \sigma \mapsto \varphi_i(\sigma).$$

So Ψ is a cont. bijection $D \to X$, hence by Prop, τ_Ψ is a Polish refinement of τ with the same Borel sets.

Claim: B is τ_Ψ-clopen.

pf(c): Check $\Psi^{-1}(B) = N_0 \cap D$ open in D
$\Psi^{-1}(X \setminus B) = N_1 \cap D$ open in D.

So B & $X \setminus B$ both τ_Ψ-open. ■(c) ■(Thm)

So we can turn one Borel set into a clopen set. Why stop there? But first, an amalgamation lemma:

Lemma: Suppose that (X, τ) is Polish, and that we have a sequence of Polish refinements $\tau_n \supseteq \tau$. Then there is a further Polish τ^* s.t. $\forall n\ \tau^* \supseteq \tau_n$. [Moreover, if $\forall n\ \mathcal{B}(\tau_n) = \mathcal{B}(\tau)$, then $\mathcal{B}(\tau^*) = \mathcal{B}(\tau)$].

(4) pf: Consider the Polish space $\prod_n (\mathbb{X}, \tau_n)$ with underlying set $\mathbb{X}^\omega$. Consider as usual the diagonal $\Delta = \{(x, x, x, \ldots) : x \in \mathbb{X}\} \subseteq \mathbb{X}^\omega$.

Claim: Δ is closed in $\prod_n (\mathbb{X}, \tau_n)$.

pf(c): Surely Δ is closed in $\prod_n (\mathbb{X}, \tau_0)$, thus it is closed in the refinement $\prod_n (\mathbb{X}, \tau_n)$. $\blacksquare$(c)

So Δ is Polish with the subspace top from $\prod_n (\mathbb{X}, \tau_n)$. Project this top. to $\mathbb{X}$, and it works. $\blacksquare$(L)

Cor: Given a countable sequence $(B_n)_{n \in \omega}$ of Borel subsets of a Polish space, there is a Polish refinement [with the same Borel sets] rendering each B_n clopen.

Def: Given top. spaces $\mathbb{X}, \mathbb{Y}$, a function $f : \mathbb{X} \to \mathbb{Y}$ is Borel if $\forall U \subseteq \mathbb{Y}$ open, $f^{-1}(U)$ is Borel.

Thm: Suppose that $\mathbb{X}$ is Polish, $\mathbb{Y}$ is second ctbl, and that $f : \mathbb{X} \to \mathbb{Y}$ is Borel. Then there is a Polish refinement of $\mathbb{X}$ [with same Borel sets] rendering f continuous.

pf: Enumerate a base $\{V_n : n \in \omega\}$ for $\mathbb{Y}$. Refine the topology on $\mathbb{X}$ to make each $f^{-1}(V_n)$ clopen. $\blacksquare$ (Thm)

DST

Lecture 12

Standard Borel spaces

Def: A set X equipped with a σ-algebra $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a standard Borel space if there is some Polish topology on X whose Borel σ-algebra is $\mathcal{B}$.

Def: Given two std Borel spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$, a function $f: X \to Y$ is Borel if

$$\forall A \in \mathcal{B}_Y \quad f^{-1}(A) \in \mathcal{B}_X.$$

Last time: Given such Borel f, there are Polish topologies on X, Y compatible with $\mathcal{B}_X, \mathcal{B}_Y$ rendering f continuous.

Thm: Suppose that X, Y are std Borel, $A \subseteq X$ Borel, and $f: X \to Y$ Borel with $f \restriction A$ injective. Then the image $f[A] \subseteq Y$ is also Borel.

pf: Fix compatible Polish topologies rendering A clopen and f continuous. Then $f[A]$ is ω-Lusin, hence Borel. ∎(Thm)

Cor: If $f: X \to Y$ is a Borel bijection between std Borel spaces, then it is an isomorphism of std Borel spaces.

pf: By the Thm, $A \subseteq X$ is Borel iff $f[A] \subseteq Y$ is Borel. ∎(Cor)

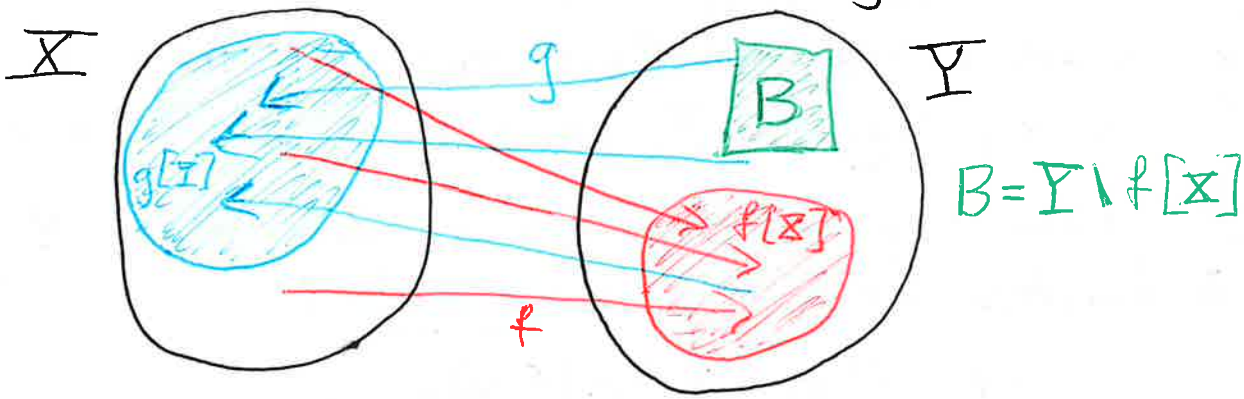
Cor: If $\mathcal{B}_0 \subseteq \mathcal{B}_1$ are two std Borel σ-algebras on X, then $\mathcal{B}_0 = \mathcal{B}_1$.

pf: Consider the identity map $(X, \mathcal{B}_1) \to (X, \mathcal{B}_0)$. ∎(Cor)

② <u>Thm</u> (Borel Schröder-Bernstein):

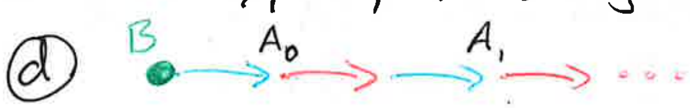
Suppose that $\mathbb{X}, \mathbb{Y}$ are std Borel spaces, and that $f: \mathbb{X} \to \mathbb{Y}$, $g: \mathbb{Y} \to \mathbb{X}$ are Borel injections. Then $\mathbb{X} \cong \mathbb{Y}$ as std Borel spaces.

<u>pf</u>: Prove SB and check that everything is Borel. ☺



Four types of f/g "orbits"

On ⓐ ⓑ ⓒ types, f is bijective $\mathbb{X} \to \mathbb{Y}$



On ⓓ type, g^{-1} is bijective $\mathbb{X} \to \mathbb{Y}$.

Put $B = \mathbb{Y} \setminus f[\mathbb{X}]$, $A_n = (gf)^n g[B]$, and $A = \bigcup_n A_n$. These are all Borel.

Build the desired Borel bijection $h: \mathbb{X} \to \mathbb{Y}$ by

$$h: x \longmapsto \begin{cases} g^{-1}(x) & \text{if } x \in A \\ f(x) & \text{if } x \notin A \end{cases}. \qquad \blacksquare(\text{Thm})$$

<u>Remark</u>: This constructed h has the bonus property: $h \subseteq f \cup \rho$, where $\rho = \{(g(y), y) : y \in \mathbb{Y}\}$

③ <u>Thm</u> (Borel isomorphism thm, Kuratowski):

Suppose that X, Y are std Borel spaces. TFAE:

$\boxed{\text{I}}$ $|X| = |Y|$

$\boxed{\text{II}}$ $X \cong Y$.

<u>pf</u>: $\boxed{\text{II}} \Rightarrow \boxed{\text{I}}$ ☺

$\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$: First observe that if X is a countable std Borel space, we must have $\mathcal{B}_X = \mathcal{P}(X)$ as singletons are Borel. So among countable spaces, any bijection is Borel.

Thus, it suffices to analyze uncountable std Borel spaces. Let us prove that any such space is isomorphic to 2^ω with usual Borel σ-alg.

Towards that end, fix an unctbl Polish space X.

<u>Claim</u>: There is a Borel injection $f: 2^\omega \hookrightarrow X$.

<u>pf(C0)</u>: Cantor. ■(C0)

<u>Claim 1</u>: There is a Borel injection $g: X \hookrightarrow 2^\omega$.

<u>pf(C1)</u>: Enumerate a base $\{\mathcal{U}_n : n \in \omega\}$ for X.

Declare $g(x)(n) = 1$ iff $x \in \mathcal{U}_n$. Now g is Borel since $\forall s \in 2^{<\omega}$ $g^{-1}(\mathcal{N}_s)$ is a Boolean combo of $\mathcal{U}_0, \dots, \mathcal{U}_{\mathrm{len}(s)-1}$. It's also injective as $\{\mathcal{U}_n\}$ separates points. ■(C1)

We are now done by Borel SB. ■(Thm)

It turns out that analyticity is "detected" by standard Borel structure:

<u>Prop</u>: Suppose that Y is Polish and $A \subseteq Y$. TFAE:

$\boxed{\text{I}}$ A is analytic.

$\boxed{\text{II}}$ $\exists$ std Borel space X
$\exists$ Borel $B \subseteq X$
$\exists$ Borel $f : X \to Y$ with $f[B] = A$.

<u>pf</u>: $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$

$\boxed{\text{II}} \Rightarrow \boxed{\text{I}}$: Fix appropriate witnesses X, B, f. Fix a compatible Polish topology on X rendering f continuous. Fix also closed $C \subseteq \omega^\omega$ and continuous $g : C \to X$ with $g[C] = B$. Then $f \circ g[C] = A$, witnessing that A is analytic. ✓ ■ (Prop)

<u>Remark</u>: Essentially the same argument establishes the equivalence of $\boxed{\text{I}}$ and $\boxed{\text{II}}$ with:

$\boxed{\text{III}}$ $\exists$ std Borel space X
$\exists$ $\boxed{\text{analytic}}$ $B \subseteq X$
$\exists$ Borel $f : X \to Y$ with $f[B] = A$.

DST

Lecture 13

Goal: Analytic subsets of Polish spaces are BP.

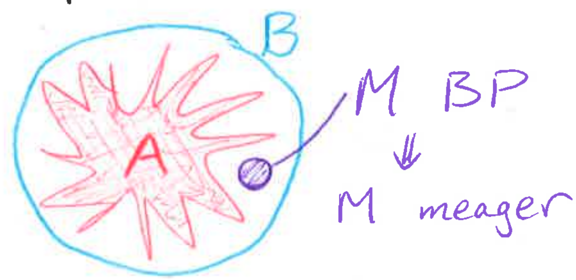
Def: Given a top space X and $A \subseteq X$, we say that $B \subseteq X$ is a BP-envelope of A if:

(a) $A \subseteq B$, and

(b) whenever $M \subseteq B \setminus A$ is BP, then M is meager.

Lemma: For any Polish X and $A \subseteq X$, there is a Borel (in fact F_σ) BP-envelope of A.

Remark: This is the "BP analogue" of witnessing the outer measure.



pf($\mathcal{L}$): Put $\mathcal{U} = \{U \subseteq X \text{ open} : A \cap U \text{ is meager}\}$.

Put $O = \bigcup \mathcal{U}$, so $A \cap O$ is still meager. $\{X$ 2nd ctbl.$\}$

[In fact, O is the maximal open set with this property.]

Find a meager F_σ set F with $A \cap O \subseteq F$.

Declare $B_0 = X \setminus O$ and finally put $B = B_0 \cup F$.

We see
$$A \subseteq (A \setminus O) \cup (A \cap O) \subseteq B_0 \cup F = B \quad \checkmark a$$

(2)

$B = B_0 \cup F$

$B_0 = X \setminus O$

<u>pf($\mathcal{L}$, cont.)</u>

<u>Claim:</u> If $M \subseteq B \setminus A$ is BP, then M is meager.

<u>pf(C):</u> Put $M_0 = M \cap B_0$, $M_1 = M \cap F$, so $M = M_0 \cup M_1$.

Since $M_1 \subseteq F$ it is meager, so it suffices to show that M_0 is meager. Towards a contradiction, suppose otherwise and localize to a non-$\emptyset$ open $\mathcal{U}$ in which M_0 is comeager. Since M_0 is disjoint from A, this means that A is meager in $\mathcal{U}$ and hence $\mathcal{U} \subseteq O$. But this means $M_0 \cap \mathcal{U} \subseteq M_0 \cap O = \emptyset$, contradicting that M_0 is comeager in $\mathcal{U}$. $\blacksquare$(C)

The claim establishes (b), completing the proof. $\blacksquare$($\mathcal{L}$)

<u>Remark:</u> If μ is a Borel probability measure on X, an outer measure argument yields a "Borel (in fact G_δ) μ-envelope."

<u>Thm</u> (Lusin-Sierpiński) In a Polish space, every analytic set is BP.

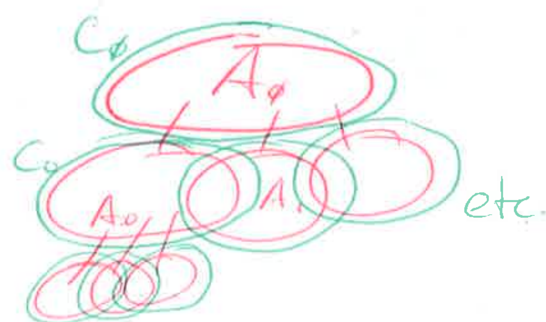
③ <u>pf</u> ($\mathcal{L}$-S). Fix Polish X and analytic $A \subseteq X$. WLOG $A \neq \emptyset$, so fix a continuous surjection $\varphi : \omega^\omega \to A$. As usual, put $A_s = \varphi[N_s]$ for $s \in \omega^{<\omega}$. Fix for each $s \in \omega^{<\omega}$ a Borel BP-envelope $B_s \supseteq A_s$. Finally, do some housekeeping, and put (recursively)

$$C_\emptyset = B_\emptyset \cap \overline{A}_\emptyset$$

$$C_{s^\frown i} = B_{s^\frown i} \cap \overline{A}_{s^\frown i} \cap C_s .$$

So we end up with a scheme $(C_s)_{s \in \omega^{<\omega}}$ s.t.

- $A_s \subseteq C_s \subseteq \overline{A}_s$
- C_s is a Borel BP-envelope of A_s
- $C_{s^\frown i} \subseteq C_s$.



Define a Borel set $C \subseteq X$ by $C = \bigcap_{n \in \omega} \bigcup_{s \in \omega^n} C_s$

<u>Claim 0:</u> $A \subseteq C$

<u>pf(C0):</u> Each $A_s \subseteq C_s$ ■ (C0)

<u>Claim 1:</u> $C \setminus A$ is meager.

<u>pf(C1):</u> We know that $A = \bigcup_{\sigma \in \omega^\omega} \bigcap_{n \in \omega} A_{\sigma \restriction n}$

$$= \bigcup_{\sigma \in \omega^\omega} \bigcap_{n \in \omega} C_{\sigma \restriction n}.$$

So if $x \in C \setminus A$, there must be some $s \in \omega^{<\omega}$ s.t.

- $x \in C_s$
- $\forall i \in \omega \;\; x \notin C_{s^\frown i}$.

 pf ($\mathcal{L}$-S, cont.): pf (C1, cont.):

In other words, $C \setminus A \subseteq \bigcup_{s \in \omega^{<\omega}} \left(C_s \setminus \bigcup_{i \in \omega} C_{s^\frown i} \right)$.

Subclaim: Each $C_s \setminus \bigcup_i C_{s^\frown i}$ is meager.

pf (s.c.): It is a BP subset of $C_s \setminus A_s$ ■ (s.c.)

■ (C1)

■ ($\mathcal{L}$-S)

Another perspective: If $\mathbb{X}$ is a std Borel space, $A \subseteq \mathbb{X}$ analytic, then for $\boxed{\text{any}}$ compatible Polish topology τ on $\mathbb{X}$, A is τ-BP.

I.e., analytic sets are "ω-universally Baire." or "globally Baire."

The same proof yields:

Thm (Lusin): If $\mathbb{X}$ is std Borel, $A \subseteq \mathbb{X}$ analytic, then for $\boxed{\text{any}}$ Borel probability measure μ on $\mathbb{X}$, A is μ-measurable.

I.e., analytic sets are "universally measurable."

DST

Lecture 14

The $\mathbb{G}_0$ dichotomy (part 1)

Def: Given graphs G on X and H on Y, a function $\varphi : X \to Y$ is a homomorphism (from G to H) if $\forall x_0, x_1 \in X \ (x_0 \, G \, x_1 \Rightarrow \varphi(x_0) \, H \, \varphi(x_1))$.

Equiv, $\forall B \subseteq Y$ (B is H-indep $\Rightarrow \varphi^{-1}(B)$ is G-indep)

Thm (Kechris-Solecki-Todorcevic, 1999):
Suppose that G is an analytic graph on a Hausdorff space X. Exactly one:

$\boxed{\text{I}}$ $\chi_B(G) \leq \aleph_0$, i.e., there is a cover of X by countably many G-indep Borel sets.

$\boxed{\text{II}}$ There is a continuous hom from $\mathbb{G}_0$ to G.

Remarks:

ⓐ We already know $\boxed{\text{I}} \Rightarrow \neg \boxed{\text{II}}$ as $\mathbb{G}_0$-indep Borel sets are meager. Just need $\boxed{\text{I}} \vee \boxed{\text{II}}$.

ⓑ We shall assume $X = \omega^\omega$. This quickly implies the general case.

ⓒ We follow Ben Miller's proof of $\boxed{\text{I}} \vee \boxed{\text{II}}$.

Today we "set the stage," and will finish the proof next lecture.

(2) Let's assume $G \neq \emptyset$ and fix a continuous function $\Theta : \omega^\omega \to \omega^\omega \times \omega^\omega$ with $\Theta[\omega^\omega] = G$.

We are trying to build a continuous function $\varphi : 2^\omega \to \omega^\omega$ so that $x \, G_0 \, y \Rightarrow \varphi(x) \, G \, \varphi(y)$.

A typical G_0-edge looks like
$$x = s_k{}^\frown 0^\frown z$$
$$y = s_k{}^\frown 1^\frown z.$$

The key idea is to use this z to "code" a G-edge between $\varphi(x)$ and $\varphi(y)$. More precisely, as we build φ we also build (for $k \in \omega$) continuous $\gamma_k : 2^\omega \to \omega^\omega$ s.t.

$$\left.\begin{matrix} x = s_k{}^\frown 0^\frown z \\ y = s_k{}^\frown 1^\frown z \end{matrix}\right\} \Longrightarrow \Theta(\gamma_k(z)) = (\varphi(x), \varphi(y)).$$

This ensures $\varphi(x) \, G \, \varphi(y)$.

As always, we realize φ and these γ_k as limits of monotone functions $2^{<\omega} \to \omega^{<\omega}$.

Def: For $n \in \omega$, an <u>$(n$-$)$ approximation</u> is a tuple
$$a = (n^a, f^a, (g_k^a)_{k<n}) \text{ where:}$$

- $n^a = n$
- $f : 2^n \to \omega^n$
- $g_k : 2^{n-(k+1)} \to \omega^n$

<u>Ex:</u> $n = 7$, $k = 2$

$$x = \overbrace{\_\ \_}^{s_2}\ \underline{0}\ \overbrace{\_\ \_\ \_\ \_}^{7-(2+1)}$$

(3) <u>Def</u>: Given an approx $a=(n, f, (g_k))$, a <u>realization</u> of a is a tuple $\alpha = (n, \varphi^\alpha, (\gamma_k^\alpha)_{k<n})$

with $\varphi^\alpha : 2^n \to \omega^\omega$

$\gamma_k^\alpha : 2^{n-(k+1)} \to \omega^\omega$

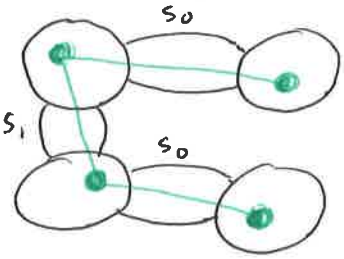
satisfying (1) $\forall s \in 2^n \;\; \varphi^\alpha(s) \in \mathcal{N}_{f(s)}$

(2) $\forall t \in 2^{n-(k+1)} \;\; \gamma_k^\alpha(t) \in \mathcal{N}_{g_k(t)}$

(3) $\Theta(\gamma_k^\alpha(t)) = (\varphi^\alpha(s_k ⌢ 0 ⌢ t), \; \varphi^\alpha(s_k ⌢ 1 ⌢ t))$.

A typical 2-approx:

A typical realization:



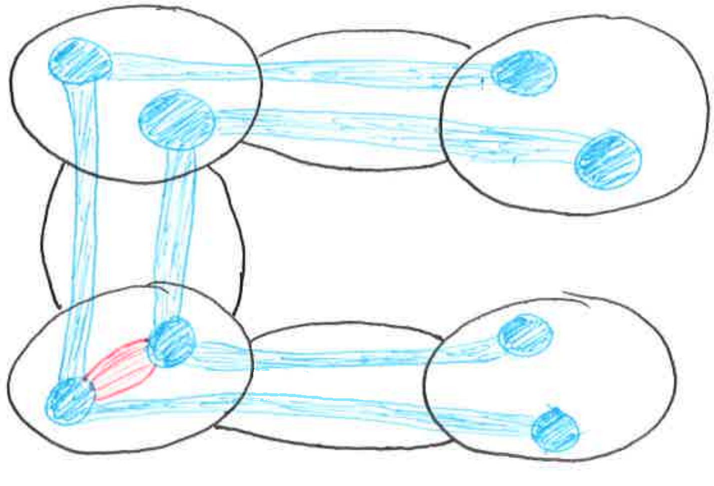
What should it mean for an $(n+1)$-approx to <u>extend</u> a given n-approx?

<u>Def</u>: For an approx $a = (n, f^a, (g_k^a))$, such a <u>one-step extension</u> is $b = (n+1, f^b, (g_k^b)_{k<n+1})$ s.t.

- $\forall s \in 2^n \;\; \forall i \in 2 \quad f^a(s) \sqsubseteq f^b(s ⌢ i)$
- $\forall k<n \;\; \forall t \in 2^{n-(k+1)} \;\; \forall i \in 2 \quad g_k^a(t) \sqsubseteq g_k^b(t ⌢ i)$
- NEW $\;\; g_n^b : \emptyset \mapsto$ some element of ω^{n+1}

a

b



NEW G_0-edges to worry about at s_2

①

DST

Lecture 15

The $\mathbb{G}_0$ dichotomy (part 2)

Def: Given $Y \subseteq \omega^\omega$, a Y-realization is a realization $\alpha = (n, \varphi, (\gamma_k))$ with $\varphi[2^n] \subseteq Y$.

Def: A set $Y \subseteq \omega^\omega$ is a G-kernel if $\forall n \in \omega$ for all n-approx a and $s \in 2^n$, the [analytic] set

$$A(a, s, Y) = \{\varphi^\alpha(s) : \alpha \text{ a } Y\text{-realization of } a\}$$

is either empty or has a G-edge. [i.e., is NOT G-indep]

Lemma: If G is an analytic graph on ω^ω, then there is a Borel G-kernel $Y \subseteq \omega^\omega$ s.t. $\omega^\omega \setminus Y$ is the union of ctbly many Borel G-indep sets. I.e., $\chi_B(G \restriction (\omega^\omega \setminus Y)) \leq \aleph_0$.

pf (L): HW! ☺ (L)

Def: Given two n-realizations $\alpha_0 = (n, \varphi^{\alpha_0}, (\gamma_k^{\alpha_0}))$
$\alpha_1 = (n, \varphi^{\alpha_1}, (\gamma_k^{\alpha_1}))$
with $\varphi^{\alpha_0}(s_n) \ G \ \varphi^{\alpha_1}(s_n)$, their amalgamation is any $\alpha_0 \oplus \alpha_1 = \beta = (n+1, \varphi^\beta, (\gamma_k^\beta)_{k<n+1})$ s.t.

- $\varphi^\beta : s^\frown i \mapsto \varphi^{\alpha_i}(s)$
- $\gamma_k^\beta : t^\frown i \mapsto \gamma_k^{\alpha_i}(t)$ for $k<n$
- $\gamma_n^\beta : \emptyset \mapsto z$ with $\Theta(z) = (\varphi^{\alpha_0}(s_n), \varphi^{\alpha_1}(s_n))$

Such $\alpha_0 \oplus \alpha_1$ is an $(n+1)$-realization.

② pf (KST): As discussed, we want $\boxed{I} \vee \boxed{II}$

By our Lemma, fix a Borel G-kernel $Y \subseteq \omega^\omega$ with $\chi_B(G \restriction (\omega^\omega \setminus Y)) \leq \aleph_0$. If $Y = \emptyset$, then certainly $\boxed{I}$ holds. So assume $Y \neq \emptyset$, working towards $\boxed{II}$.

Def: An n-approx a is Y-realized if it has a Y-realization. I.e., $\forall s \in 2^n$ $A(a, s, Y) \neq \emptyset$.

Remarks:

ⓐ Since Y is a G-kernel, the above implies

$\forall s \in 2^n$ $A(a, s, Y)$ is NOT G-indep.

ⓑ As $Y \neq \emptyset$, the unique 0-approx $a = (0, \emptyset \mapsto \emptyset)$ is Y-realized by any $\alpha = (0, \emptyset \mapsto y)$ with $y \in Y$.

Splitting Lemma:

Every Y-realized approx has a Y-realized one-step ext.

pf (S.L.):

Fix a Y-realized n-approx a. We know that $A(a, s_n, Y)$ is non-$\emptyset$ and has a G-edge. Fix Y-realizations α_0, α_1 of a with

$$\varphi^{\alpha_0}(s_n) \; G \; \varphi^{\alpha_1}(s_n).$$

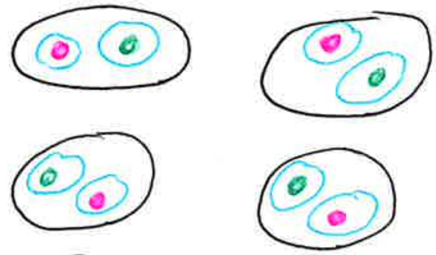
Then any amalgamation $\alpha_0 \oplus \alpha_1$ Y-realizes a one-step extension of a. More formally, if $\beta = \alpha_0 \oplus \alpha_1$, our one-step ext $b = (n+1, f^b, (g_k^b))$

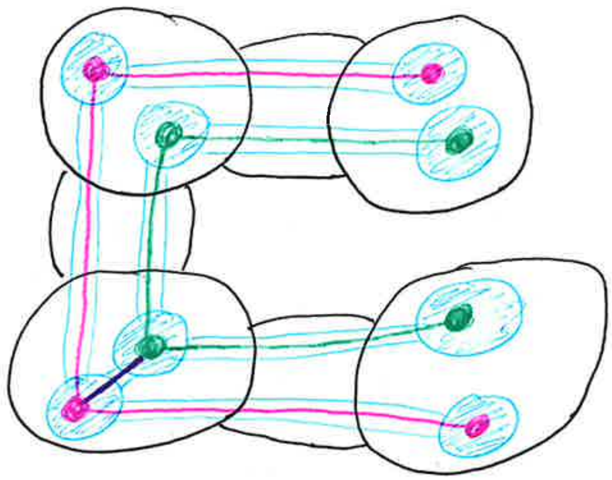
$$f^b : s \mapsto \varphi^\beta(s) \restriction (n+1)$$

$$g_k^b : t \mapsto \gamma_k^\beta(t) \restriction (n+1).$$

■ (S.L.)

③ <u>Cartoony aside</u>: Let's contrast S.L.s at stage $2 \to 3$.

Perfect set thm: 

G_0 : 

2-approx a
$\downarrow$
2-realizations α_0, α_1
$\downarrow$
3-realization $\alpha_0 \oplus \alpha_1$
$\downarrow$
3-approx b

<u>pf (KST, cont.)</u>:

Upon iterating S.L., we obtain a coherent sequence

$$a_n = (n, f^{a_n}, (g_k^{a_n})_{k<n})$$

of $\mathbb{I}$-realized approximations.

By monotonicity, we may define continuous functions

$$\varphi : 2^\omega \to \omega^\omega \qquad \gamma_k : 2^\omega \to \omega^\omega$$

by $\quad \varphi = \lim_n f^{a_n} \qquad \gamma_k = \lim_n g_k^{a_n}$

There's only one remaining detail...

(4) pf (KST, cont.):

Claim: φ is a hom from $\mathbb{G}_0$ to $\mathbb{G}$.

pf(C): We show $\forall k \in \omega$ and $z \in 2^\omega$ that

$$\Theta(\gamma_k(z)) = (\varphi(s_k{}^\frown 0^\frown z), \varphi(s_k{}^\frown 1^\frown z)).$$

Since Θ is continuous, its "graph"

$$\{(x, y_0, y_1) \in (\omega^\omega)^3 : \Theta(x) = (y_0, y_1)\}$$

is closed in $(\omega^\omega)^3$. It thus suffices to show that any open $\mathcal{U} \subseteq (\omega^\omega)^3$ with

$$(\gamma_k(z), \varphi(s_k{}^\frown 0^\frown z), \varphi(s_k{}^\frown 1^\frown z)) \in \mathcal{U}$$

has non-$\emptyset$ intersection with Θ's "graph." Fix $n > k$ sufficiently large s.t.

$$\mathcal{N}_{\gamma_k(z)\restriction n} \times \mathcal{N}_{\varphi(s_k{}^\frown 0^\frown z)\restriction n} \times \mathcal{N}_{\varphi(s_k{}^\frown 1^\frown z)\restriction n} \subseteq \mathcal{U}.$$

Put $t = z\restriction(n-(k+1))$, so this becomes

$$\mathcal{N}_{g_k^{a_n}(t)} \times \mathcal{N}_{f^{a_n}(s_k{}^\frown 0^\frown t)} \times \mathcal{N}_{f^{a_n}(s_k{}^\frown 1^\frown t)} \subseteq \mathcal{U}.$$

Fixing a Υ-realization α of a_n, we see

$$(\gamma_k^\alpha(t), \varphi^\alpha(s_k{}^\frown 0^\frown t), \varphi^\alpha(s_k{}^\frown 1^\frown t))$$

is in the LHS, hence is in $\mathcal{U}$. By definition,

$$\Theta(\gamma_k^\alpha(t)) = (\varphi^\alpha(s_k{}^\frown 0^\frown t), \varphi^\alpha(s_k{}^\frown 1^\frown t))$$

ensuring that Θ's "graph" meets $\mathcal{U}$. ■(C)

The claim ensures that $\boxed{\text{II}}$ holds whenever $\Upsilon \neq \emptyset$, completing our proof. ■(KST)

DST

Lecture 16

Applying the $\mathbb{G}_0$ dichotomy

Example: Suppose that A is an analytic subset of a Polish space X. Consider the complete graph $K_A = \{(x,y) \in A^2 : x \neq y\}$ on A. It is an analytic graph, since $K_A = A^2 \cap (X^2 \setminus \Delta)$.

Let's consider the alternatives in the $\mathbb{G}_0$ dichotomy:

$\boxed{\text{I}}$ $\chi_B(K_A) \leq \aleph_0$. Note that K_A-indep subset of A has cardinality at most 1. We get $|A| \leq \aleph_0$.

$\boxed{\text{II}}$ There's a continuous hom $\varphi : 2^\omega \to A$ from $\mathbb{G}_0$ to K_A. How injective is φ? Given $a \in A$, $\{a\}$ is K_A-indep, so we see $\varphi^{-1}(\{a\}) \subseteq 2^\omega$ is closed and $\mathbb{G}_0$-indep. In other words, φ is a meager-to-one function from 2^ω to A.

We would like a true (one-to-one) injection. Our next tool furnishes us with a handy way to sidestep this meager amount of error.

Thm (Mycielski): Suppose that $R \subseteq 2^\omega \times 2^\omega$ is comeager. Then there is a cont. inj. $\varphi: 2^\omega \to 2^\omega$ s.t. $\forall x, y \in 2^\omega$

$$x \neq y \Rightarrow (\varphi(x), \varphi(y)) \in R.$$

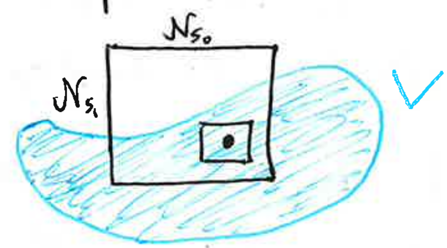
"Any comeager graph has a Cantor set clique."

Remark: This holds for comeager binary relations on any perfect Polish space. Also for larger arity.

Mycielski's theorem really boils down to:

Observation: If $V \subseteq 2^\omega \times 2^\omega$ is open dense and $s_0, s_1 \in 2^{<\omega}$, then there are extensions $t_i \sqsupseteq s_i$ with

$$\mathcal{N}_{t_0} \times \mathcal{N}_{t_1} \subseteq V.$$



pf (Obs): Fix $(x,y) \in (\mathcal{N}_{s_0} \times \mathcal{N}_{s_1}) \cap V$. Find large $n \in \omega$ with $\mathcal{N}_{x \restriction n} \times \mathcal{N}_{y \restriction n} \subseteq V$. Then put $t_0 = x \restriction n$, $t_1 = y \restriction n$. ■ (obs)

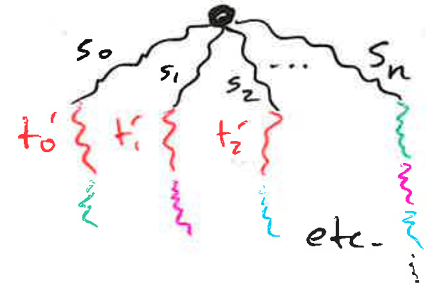
Let's promote this observation into:

Lemma: Suppose that $(s_i)_{i<n} \in 2^{<\omega}$ and $V \subseteq 2^\omega \times 2^\omega$ is open dense. Then there are extensions $t_i \sqsupseteq s_i$ such that

$$i \neq j \Rightarrow \mathcal{N}_{t_i} \times \mathcal{N}_{t_j} \subseteq V.$$

Moreover, we may ensure all t_i have the same length.

(3) pf ($\mathcal{L}$, sketch): Induct on n.



Given $s_0, \ldots, s_n \in 2^{<\omega}$, first find for $i<n$ $t_i' \sqsupseteq s_i$ that work among themselves. Then, for $i<n$ apply the observation iteratively to get extensions $t_i \sqsupseteq t_i'$, $t_n \sqsupseteq s_n$ with $\mathcal{N}_{t_i} \times \mathcal{N}_{t_n} \subseteq V$ and $\mathcal{N}_{t_n} \times \mathcal{N}_{t_i} \subseteq V$. ∎($\mathcal{L}$, sketch)

pf (Mycielski): Fix open dense $(\mathcal{U}_n)_{n\in\omega}$ with $\bigcap_n \mathcal{U}_n \subseteq R$. Write $V_n = \bigcap_{m<n} \mathcal{U}_m$ [so $V_0 = 2^\omega \times 2^\omega$]. Then V_n is still open dense with $\bigcap_n V_n \subseteq R$, but also $V_{n+1} \subseteq V_n$. We shall recursively construct functions $f_n : 2^n \to 2^{<\omega}$ s.t.

- $\forall s,t \in 2^n$ $f_n(s)$ and $f_n(t)$ have the same length ($\geq n$)
- $s \neq t \Rightarrow f_n(s) \neq f_n(t)$
- $s \neq t \Rightarrow \mathcal{N}_{f_n(s)} \times \mathcal{N}_{f_n(t)} \subseteq V_n$
- $f_{n+1}(s^\frown i) \sqsupseteq f_n(s)^\frown i$ monotone +

Stage 0: $f_0 : \emptyset \mapsto \emptyset$ ☺

Stage n+1: Given $f_n : 2^n \to 2^{<\omega}$ as above, consider the strings $f_n(s)^\frown i$ for $s \in 2^n$, $i \in 2$. The Lemma grants ext'ns $f_{n+1}(s^\frown i) \sqsupseteq f_n(s)^\frown i$ such that $s^\frown i \neq t^\frown j \Rightarrow \mathcal{N}_{f_{n+1}(s^\frown i)} \times \mathcal{N}_{f_{n+1}(t^\frown j)} \subseteq V_{n+1}$.

The remaining conditions are easy to check, and the recursive construction is complete.

pf (Mycielski, cont.): Now put $\varphi = \lim_n f_n$.

Claim: This φ works.

pf (C): Continuous? ☺ ✓

Injective? Given $x \neq y \in 2^\omega$, we may find large $n \in \omega$ with $x \restriction n \neq y \restriction n$. This means $f_n(x\restriction n) \neq f_n(y \restriction n)$ and thus $\varphi(x) \neq \varphi(y)$. ✓

$(\varphi(x), \varphi(y)) \in R$? Since $\bigcap_n V_n \subseteq R$, it suffices to show that $(\varphi(x), \varphi(y)) \in V_n$ for large n. Given $x \neq y$, consider n large enough so that $x \restriction n \neq y \restriction n$. Then

$$\varphi(x) \in \mathcal{N}_{f_n(x \restriction n)}$$
$$\varphi(y) \in \mathcal{N}_{f_n(y \restriction n)}$$

and thus $(\varphi(x), \varphi(y)) \in V_n$ as desired. ✓ ■(C)

■(Mycielski)

Cor: Suppose that X is std Borel and that $\Theta : 2^\omega \to X$ is a Borel, meager-to-one function. Then there is continuous $\varphi : 2^\omega \to 2^\omega$ so that the composition $\Theta \circ \varphi : 2^\omega \to X$ is injective.

pf (Cor): Consider the Borel set $R \subseteq 2^\omega \times 2^\omega$ defined by $(x,y) \in R$ iff $\Theta(x) = \Theta(y)$. Every vertical section of R is meager, thus by Kuratowski-Ulam R itself is meager. Applying Mycielski to $(2^\omega \times 2^\omega) \setminus R$ yields continuous $\varphi : 2^\omega \to 2^\omega$ s.t. $x \neq y \Rightarrow (\varphi(x), \varphi(y)) \notin R$. I.e., $x \neq y \Rightarrow \Theta \circ \varphi(x) \neq \Theta \circ \varphi(y)$. ■(Cor)

DST

Lecture 17

Last time (Suslin, 1917): Suppose that A is analytic.

Exactly one:

$\boxed{\text{I}}$ A is countable

$\boxed{\text{II}}$ There is a continuous injection $2^\omega \hookrightarrow A$.

pf: Apply the $\mathbb{G}_0$ dichotomy to K_A. If $\chi_B(K_A) \leq \aleph_0$, then $\boxed{\text{I}}$ holds. Else, there is a cont. hom from $\mathbb{G}_0$ to K_A. This is a meager-to-1 function $2^\omega \to A$. Mycielski allows us to thin this to a cont inj $2^\omega \hookrightarrow A$, granting $\boxed{\text{II}}$. ∎ (Suslin, last time)

This suggests a powerful threefold path, pioneered by Ben Miller:

Step 1: Analyze some analytic graph using the $\mathbb{G}_0$ dichotomy.

Step 2: Prove that some annoying set is meager, typically using Kuratowski-Ulam.

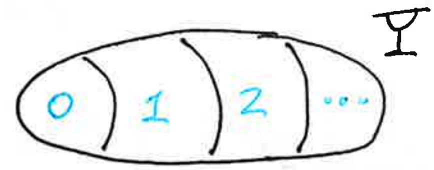
Step 3: Sidestep this annoying set using Mycielski.

Let's next apply this method to prove perfect set theorems for QUOTIENTS.

Def: A set is coanalytic if its complement is analytic.

② **Thm** (Silver, 1980): Suppose that Y is a Polish space and that E is a coanalytic equivalence relation on Y. Exactly one:

$\boxed{\mathrm{I}}$ E has ctbly many classes



$\boxed{\mathrm{II}}$ $\exists$ cont inj $\varphi : 2^\omega \hookrightarrow Y$ s.t. $x_0 \neq x_1 \Rightarrow \varphi(x_0) \not{E}\, \varphi(x_1)$.

pf (Silver): Certainly $\boxed{\mathrm{I}} \Rightarrow \neg \boxed{\mathrm{II}}$.

To establish $\boxed{\mathrm{I}} \vee \boxed{\mathrm{II}}$, we apply the G_0 dichotomy to the analytic graph $G = Y^2 \setminus E$. That is,

$$y_0 \, G \, y_1 \quad \text{iff} \quad y_0 \not{E}\, y_1 .$$

Case I: $\chi_B(G) \leq \aleph_0$. Each color class is contained in a single E-class. So then E has ctbly many classes, establishing $\boxed{\mathrm{I}}$. ✓

Case II: $\exists$ cont hom $\Theta : 2^\omega \to Y$ from G_0 to G.

Define a pull-back eq. rel. F on 2^ω by

$$x_0 \, F \, x_1 \quad \text{iff} \quad \Theta(x_0) \, E \, \Theta(x_1).$$

This F is the preimage of E under the continuous function $(x_0, x_1) \mapsto (\Theta(x_0), \Theta(x_1))$. So F is coanalytic and thus BP.

Claim: $F \subseteq 2^\omega \times 2^\omega$ is meager.

pf(C): For each $x \in 2^\omega$, $[\Theta(x)]_E$ is G-indep. Thus $F_x = [x]_F = \Theta^{-1}([\Theta(x)]_E)$ is BP G_0-indep. So each F_x is meager, and F is meager by K-U.

$\blacksquare$ (C)

③ <u>pf (Silver, cont.)</u>:

Since $F \subseteq 2^\omega \times 2^\omega$ is meager, Mycielski grants a continuous function $\varphi : 2^\omega \to 2^\omega$ s.t.

$$x_0 \neq x_1 \Rightarrow \varphi(x_0) \not F \ \varphi(x_1)$$
$$\Rightarrow \Theta \circ \varphi(x_0) \not E \ \Theta \circ \varphi(x_2)$$

So $\Theta \circ \varphi$ works for $\boxed{\text{II}}$ ✓ ■ (Silver)

In other words, CH is "true for quotients by Borel (or even coanalytic) equivalence relations."

<u>Digression</u>: What about quotients by analytic equivalence relations?

Typically, CH **FAILS** in this context!

<u>Recall</u>: Binary relations on ω may be encoded as elements of $2^{\omega \times \omega} \cong 2^\omega$ via $R \mapsto x_R$ with

$$x_R : (m,n) \longmapsto \begin{cases} 1 & \text{if } m R n \\ 0 & \text{if not.} \end{cases}$$

Under this encoding, the set LO of (codes for) linear orders is a closed, hence Polish, subset of $2^{\omega \times \omega}$.

The set of (codes for) <u>ill-founded</u> orders is analytic:

$$x_R \text{ is ill-fdd iff } \exists \sigma \in \omega^\omega \ \forall n \in \omega$$
$$\sigma(n) \neq \sigma(n+1) \text{ and } x_R(\sigma(n+1), \sigma(n)) = 1.$$

Thus, the set WO of (codes for) wellorders on ω is a coanalytic subset of LO.

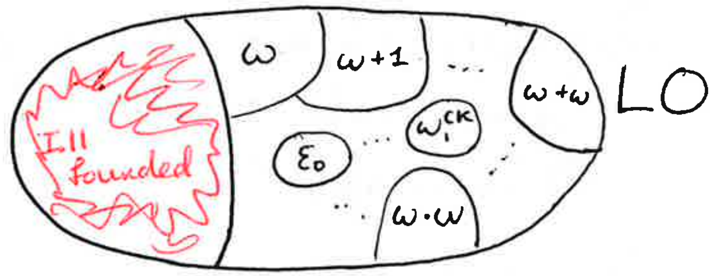
④ Define an equivalence relation E on LO by

$$x_R \, E \, x_S \text{ iff } \begin{cases} x_R \,\&\, x_S \text{ both ill-founded, or} \\ x_R \text{ is isomorphic to } x_S \end{cases}$$

<u>Claim</u>: E is analytic.

<u>pf(C)</u>: We already know ill-foundedness is analytic. For the second clause, "x_R is isomorphic to x_S" may be written $\exists \sigma \in \omega^\omega$ (σ does the isomorphism). $\blacksquare$(C)

On LO, this E has one garbage class, and also one interesting class for each ctbly infinite ordinal.



This means E has exactly $\aleph_1$-many classes, irrespective of the value of the continuum $2^{\aleph_0}$.

$\boxed{\text{HW?}}$ This E satisfies neither $\boxed{\text{I}}$ nor $\boxed{\text{II}}$ of Silver.

<u>Remarks</u>:

ⓐ If E is analytic and $|X/E| \geq \aleph_2$, then $\boxed{\text{II}}$ from Silver holds...

ⓑ If E is the orbit eq rel of a continuous action of a Polish group on a Polish space, it is STILL OPEN whether the Silver dichotomy holds of E.

$\boxed{\text{Topological Vaught Conjecture}}$

DST

Lecture 18

Algebraic topology

<u>Thm</u> (Shelah, 1988): No Peano continuum has fundamental group isomorphic to $(\mathbb{Q};+)$.

We need millions of definitions.

- Throughout, fix a compact metric space (Y,d)
- Denote by I the unit interval $[0,1] \subseteq \mathbb{R}$.

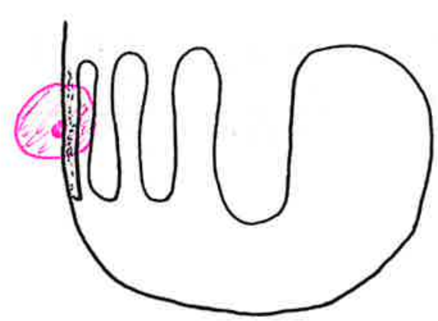
<u>Defs</u>:

- A <u>path</u> is a continuous function $f: I \to Y$. We'll call it a "path from $f(0)$ to $f(1)$."
- Y is <u>path-connected</u> (PC) if there is a path between any two points.
- Y is <u>locally path-connected</u> (LPC) if every point has a nbhd base of PC sets.
- Y is a <u>Peano continuum</u> if it's PC + LPC.

<u>Examples</u>:

(a) LPC but $\neg$PC

(b) PC but $\neg$LPC



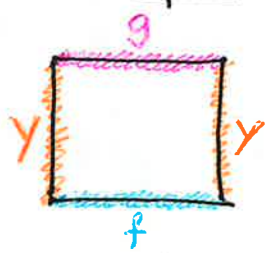
Def: For $y \in Y$ a <u>(y-) loop</u> is a path f satisfying $f(0) = f(1) = y$.

Remark: We may place the <u>uniform/sup metric</u> on the set of paths: $\delta(f, g) = \sup\{d(f(i), g(i)) : i \in I\}$. This metric is separable and complete, yielding a <u>Polish space of paths</u> and a (closed) <u>Polish subspace of y-loops</u>.

Def: A <u>(y-) homotopy</u> is a continuous function $\varphi : I^2 \to Y$ s.t. $\forall j \in I$ $\varphi(0, j) = \varphi(1, j) = y$.

The space of y-homotopies (w/ sup metric) is Polish.

Def: Two y-loops f, g are <u>(y-) homotopic</u> if there is a y-homotopy φ with
$$\varphi : (i, 0) \longmapsto f(i)$$
$$\varphi : (i, 1) \longmapsto g(i).$$



We denote this by $f \sim g$.

Prop: Homotopy is an analytic equivalence relation.

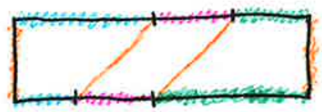
pf: The maps sending a homotopy φ to the loops $\begin{cases} i \longmapsto \varphi(i, 0) \\ i \longmapsto \varphi(i, 1) \end{cases}$ are continuous.

$\blacksquare$ (Prop)

③ **An operation:**

Given two paths f, g with $f(1) = g(0)$, their concatenation is the (normalized) path

This operation induces a group operation on homotopy classes of y-loops. This is the (y-) fundamental group.

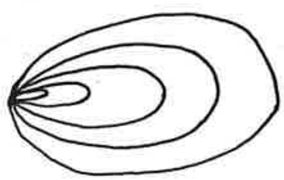
Assoc: 

Def: A (y-) loop is null-homotopic if it is homotopic to the constant y-loop, $\ell_y : i \mapsto y$.

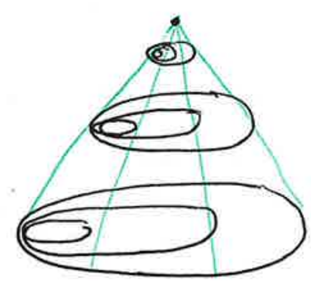
Def: Y is semi-locally simply connected (SLSC) if there is a base $\mathcal{U}$ for the topology so that $\forall U \in \mathcal{U}$ every loop $f : I \to U$ is null-homotopic in Y

Examples:

ⓐ not SLSC



ⓑ SLSC



Prop: If Y is a SLSC Peano continuum, then its fundamental group is finitely generated.

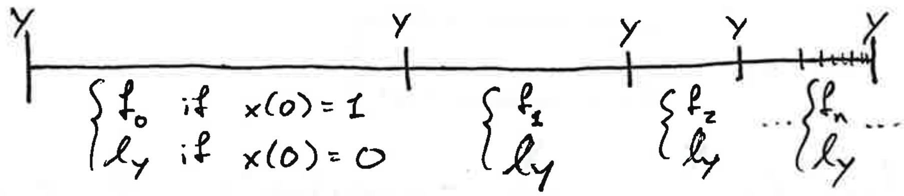
pf (sketch): By compactness, cover Y by finite $\mathcal{C}$ s.t. $\forall U, V \in \mathcal{C}$
- U open and PC
- $U \cap V \neq \emptyset \Rightarrow U \cup V$ SC in Y.

This yields a finite graph whose fund. grp surjects onto that of Y. ∎ (sketch)

(4) <u>Prop</u>: If Y is a non-SLSC Peano continuum, then its fundamental group has cardinality $2^{\aleph_0}$.

<u>pf</u>: By compactness, we may fix $y \in Y$ and a sequence $(f_n)_{n \in \omega}$ of non-null-homotopic y-loops satisfying $\delta(f_n, l_y) < \varepsilon_n \to 0$.

With each $x \in 2^\omega$ we associate a y-loop f_x:



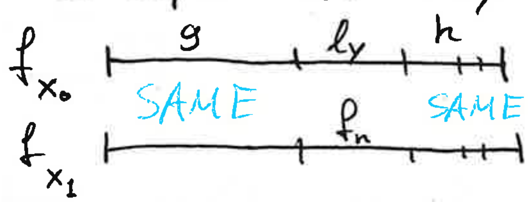
Note that each f_x is indeed continuous as $\varepsilon_n \to 0$.

Moreover, the map $x \mapsto f_x$ is continuous.

Thus, homotopy pulls back to an analytic (BP) eq. rel on 2^ω: $x_0 \, E \, x_1$ iff $f_{x_0} \sim f_{x_1}$.

<u>Claim</u>: Each E-class is $\mathbb{G}_0$-indep.

<u>pf(C)</u>: If $x_0 \, \mathbb{G}_0 \, x_1$, then x_0 & x_1 differ at exactly one input n. Say $x_0(n) = 0$, $x_1(n) = 1$.



In fund grp, $[f_{x_0}] = [g][l_y][h]$ and $[f_{x_1}] = [g][f_n][h]$.

Thus, $l_y \not\sim f_n$ implies $f_{x_0} \not\sim f_{x_1}$ as desired. ■(C)

So E is meager by K-U, and Mycielski yields a Cantor set of p-w non-homotopic loops.

■(Prop)

Shelah's thm (and more!) immediately follows.

DST
Lecture 19

Recently, we've seen some CH-ish phenomena:

- "CH for analytic sets"
- "CH for quotients by coanalytic eq rels"

Let's start investigating quotients more systematically.

Question: Given eq rels E on X and F on Y, what should "X/E injects into Y/F" mean from the perspective of X and Y?

We want a function $f: X \to Y$ such that

(a) $x_0 \, E \, x_1 \Rightarrow f(x_0) \, F \, f(x_1)$ [well defined on X/E]

(b) $f(x_0) \, F \, f(x_1) \Rightarrow x_0 \, E \, x_1$ [injective]

Def: A reduction from E on X to F on Y is a function $f: X \to Y$ satisfying

$$x_0 \, E \, x_1 \text{ iff } f(x_0) \, F \, f(x_1).$$

I.e., it is a hom from $(E, \not E)$ to $(F, \not F)$.

Notation: Given E, F on std Borel X, Y (resp.), we write $E \leq_B F$ if there is a Borel reduction from E to F.

2 In this language, we may reinterpret Silver's thm:

Given coanalytic E on std Borel X, exactly one:

$\boxed{\text{I}}$ $E \leq_B \Delta_\omega$ Recall: $\Delta_Y = \{(y,y) : y \in Y\}$

$\boxed{\text{II}}$ $\Delta_{2^\omega} \leq_B E$.

So the "first" $(\omega+2)$-many Borel/coanalytic eq rels are:

$$\Delta_0 <_B \Delta_1 <_B \Delta_2 <_B \cdots <_B \Delta_\omega \underset{\text{GAP}}{<_B} \Delta_{2^\omega} <_B ???$$

To continue this picture, we must develop tools (beyond classical cardinality) that PRECLUDE Borel reductions.

Ergodicity

<u>Def</u>: Suppose that E is an eq rel on std Borel space X. Given a compatible Polish top τ on X, we say that E is <u>$(\tau\text{-})$ generically ergodic</u> if every E-invariant τ-BP set is either τ-meager or τ-comeager.

<u>Recall</u>: A is <u>E-invariant</u> if $(x \in A \wedge x \, E \, x') \Rightarrow x' \in A$.

<u>Def</u>: If we instead specify a Borel prob measure μ on X, we say that E is <u>$(\mu\text{-})$ ergodic</u> if every E-invariant μ-meas set is either μ-null or μ-conull.

③ <u>Prop</u>: $\mathbb{E}_0$ on 2^ω is generically ergodic (in usual top).

<u>pf</u>: Suppose that $A \subseteq 2^\omega$ is BP and $\mathbb{E}_0$-inv.

If A is meager we are done! Else, we may localize to $\mathcal{N}_s$ in which A is comeager.

Put $n = \mathrm{len}(s)$, and note that $\forall t \in 2^n$ there is a homeomorphism φ_t flipping finitely many bits with $\varphi_t[\mathcal{N}_s] = \mathcal{N}_t$. $\mathbb{E}_0$-inv grants $\varphi_t[A] = A$.

So A is comeager in every $\mathcal{N}_t$, thus in 2^ω. ■(Prop)

<u>Remarks</u>:

ⓐ $\mathbb{E}_0$ is μ-ergodic for the coin-flip measure.

ⓑ Sim. argument works for the orbit eq rel of irrat'l rotation.

ⓒ Sim. argument works for coset eq rel $\mathbb{Q} \leq \mathbb{R}$.

Let's connect ergodicity with our discussion of reducibility.

<u>Prop</u>: Suppose that E is an eq. rel on Polish X. TFAE:

$\boxed{\text{I}}$ E is generically ergodic,

$\boxed{\text{II}}$ Given any std Borel Y and BP-meas $f: X \to Y$ that is a hom from E to Δ_Y, $\exists y \in Y$ s.t. $f^{-1}(\{y\})$ is comeager.

<u>Remark</u>: The analogous result holds of μ-ergodicity.

<u>pf</u>: $\boxed{\text{II}} \Rightarrow \boxed{\text{I}}$: Let $Y = \{y_0, y_1\}$ be a two-pt space.

Given $A \subseteq X$ BP E-invariant, consider

$$f : x \longmapsto \begin{cases} y_0 & \text{if } x \in A \\ y_1 & \text{if } x \notin A. \end{cases}$$

✓

④ pf (Prop, cont.)

$\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$: Fix a countable algebra $\mathcal{A} \subseteq \mathcal{B}_Y$ that generates $\mathcal{B}_Y$ as a σ-algebra. Then for all $A \in \mathcal{A}$ we know $f^{-1}(A)$ is BP and E-invariant, so is either meager or comeager by ergodicity. Put $\mathcal{A}_{BIG} = \{A \in \mathcal{A} : f^{-1}(A) \text{ is comeager}\}$

Claim 0: $\bigcap \mathcal{A}_{BIG} \neq \emptyset$.

pf(C0): Note that $f^{-1}(\bigcap \mathcal{A}_{BIG}) = \bigcap \{f^{-1}(A) : A \in \mathcal{A}_{BIG}\}$ is comeager and thus non-$\emptyset$. It follows that $\bigcap \mathcal{A}_{BIG} \neq \emptyset$ as well. ■(C0)

Claim 1: $|\bigcap \mathcal{A}_{BIG}| \leq 1$.

pf(C1): Consider $y_0 \neq y_1$ in Y. Find $A \in \mathcal{A}$ with $y_0 \in A$, $y_1 \in Y \setminus A$. Exactly one of A, $Y \setminus A$ is in $\mathcal{A}_{BIG}$, hence at most one of y_0, y_1 survives the intersection. ■(C1)

Finally, put $\bigcap \mathcal{A}_{BIG} = \{y\}$. This choice of y makes $f^{-1}(\{y\})$ comeager as desired. ✓ ■(Prop)

Let's illustrate how this yields non-reducibility

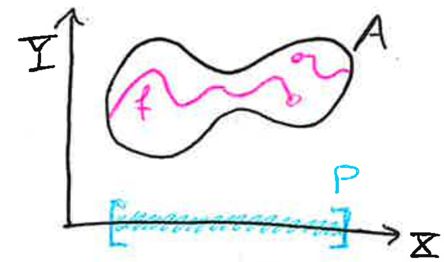
Cor: $\mathbb{E}_0 \not\leq_B \Delta_{2^\omega}$

pf: Towards a contradiction, suppose that $f : 2^\omega \to 2^\omega$ is a Borel reduction from $\mathbb{E}_0$ to Δ_{2^ω}. In particular, it is a BP-meas hom, so generic ergodicity of $\mathbb{E}_0$ grants $y \in 2^\omega$ with $f^{-1}(\{y\})$ comeager. But $f^{-1}(\{y\})$ contains at most one $\mathbb{E}_0$-class, thus is countable (hence meager). $\Rightarrow\Leftarrow$. ■(Cor)

DST

Lecture 20

A COLORFUL approach to uniformization



__Def:__ Suppose that $A \subseteq \mathbb{X} \times \mathbb{Y}$, and put $P = \mathrm{Proj}_{\mathbb{X}}[A]$. A __uniformization__ of A is $f : P \to \mathbb{Y}$ with $f \subseteq A$.

__Remark:__ Such uniformizations always exist (AC). We are interested in finding "descriptively nice" ones.

Let's re-examine (partial) functions altogether! Consider the graph G on $\mathbb{X} \times \mathbb{Y}$ defined by

$$(x_0, y_0) \, G \, (x_1, y_1) \text{ iff } x_0 = x_1 \text{ and } y_0 \neq y_1.$$

A partial function $\mathbb{X} \to \mathbb{Y}$ is exactly a G-indep set!

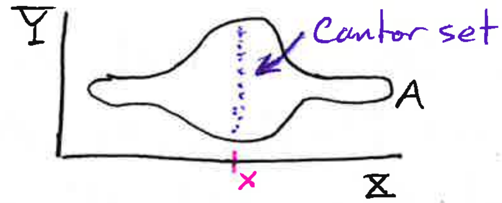
__Warm-up:__ Suppose that $\mathbb{X}, \mathbb{Y}$ are non-$\emptyset$ std Borel and that $g \subseteq \mathbb{X} \times \mathbb{Y}$ is an analytic partial function. Then there is Borel $f : \mathbb{X} \to \mathbb{Y}$ with $g \subseteq f$.

__pf (w-u):__ As discussed above, g is an analytic G-indep subset of $\mathbb{X} \times \mathbb{Y}$. Since G is analytic, g is contained in a Borel G-indep set f_0. This f_0 is a Borel partial function with Borel domain, thus may be extended to a total Borel function $f : \mathbb{X} \to \mathbb{Y}$ (since $\mathbb{Y} \neq \emptyset$). ▨ (w-u)

② **Thm** (Lusin–Novikov): Suppose that X, Y are non-$\emptyset$ Polish spaces and $A \subseteq X \times Y$ is analytic. Exactly one:

$\boxed{\text{I}}$ There are Borel functions $f_n : X \to Y$ with $A \subseteq \bigcup_n f_n$.

$\boxed{\text{II}}$ For some $x \in X$ there is continuous inj $2^\omega \hookrightarrow A_x$



pf: Define an analytic graph G_A on A as before:

$$(x_0, y_0) \; G_A \; (x_1, y_1) \text{ iff } x_0 = x_1 \text{ and } y_0 \neq y_1.$$

We apply the G_0 dichotomy to G_A.

Case I: $\chi_B(G_A) \leq \aleph_0$. Each color class $g_n \subseteq A$ is Borel in A, thus analytic in $X \times Y$. It is also a partial function by definition of G_A. By our warm-up, we may find Borel $f_n : X \to Y$ with $g_n \subseteq f_n$. Then $A = \bigcup_n g_n \subseteq \bigcup_n f_n$ as in $\boxed{\text{I}}$ ✓

Case II: There is a continuous hom $\varphi : 2^\omega \to A$ from G_0 to G Then $\mathrm{Proj}_X \circ \varphi : 2^\omega \to X$ is a cont. hom from G_0 to Δ_X as any two G_A-adjacent vertices share x-coordinates. As E_0 is the connectedness relation of G_0, it follows that $\mathrm{Proj}_X \circ \varphi$ is also a hom from E_0 to Δ_X. This seems familiar...

③ pf ($\mathcal{L}$-N, cont.):

Generic ergodicity of $\mathbb{E}_0$ grants $x \in X$ so that $C = (\mathrm{Proj}_X \circ \varphi)^{-1}(\{x\})$ is comeager in 2^ω.
Unfolding this, $C = \varphi^{-1}(\{x\} \times A_x)$. Moreover, for all $y \in A_x$ we know that $\varphi^{-1}(\{(x,y)\})$ is BP and $\mathbb{G}_0$-independent, hence is meager.
In summary,
$$\mathrm{Proj}_Y \circ \varphi : C \to A_x$$
$$c \mapsto y \text{ s.t. } \varphi(c) = (x,y)$$
is continuous and meager-to-one. Mycielski allows us to thin to a cont inj $2^\omega \hookrightarrow A_x$ as in $\boxed{\mathrm{II}}$ ✓ ■($\mathcal{L}$-N)

Cor (Lusin-Novikov uniformization theorem for Borel subsets of the plane with countable vertical sections)
Suppose that X, Y are std Borel and that $B \subseteq X \times Y$ is Borel s.t. $\forall x \in X$ $|B_x| \leq \aleph_0$. Then:

ⓐ $P = \mathrm{Proj}_X[B]$ is Borel

ⓑ B admits a Borel uniformization, i.e., a Borel function $g: P \to Y$ with $g \subseteq B$.

ⓒ In fact, there is a countable sequence $g_n : P \to Y$ of Borel uniformizations with $B = \bigcup_n g_n$.

④ pf($\mathcal{L}$NUTFBSOTPWCVS): WLOG $Y \neq \emptyset$.

We apply the previous theorem to B, observing that the countability of each B_x precludes alternative $\boxed{\text{II}}$. Thus, $\boxed{\text{I}}$ grants Borel functions $f_n : X \to Y$ with $B \subseteq \bigcup_n f_n$.

ⓐ For each $n \in \omega$, put $P_n = \mathrm{Proj}_X[B \cap f_n]$. Each P_n is a continuous injective image of a Borel set, thus is again Borel. Then $P = \mathrm{Proj}_X[B] = \bigcup_n P_n$ is also Borel ✓a

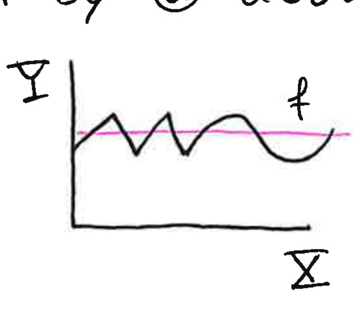
ⓑ Put $C_n = P_n \setminus \bigcup_{m<n} P_m$ and $g = \bigcup_n f_n \upharpoonright C_n$. ✓b

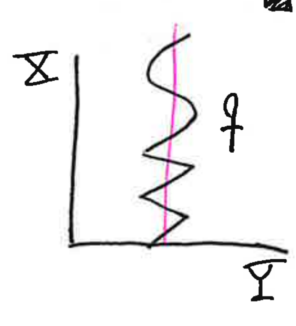
ⓒ Put $g_n = f \upharpoonright P_n \cup g \upharpoonright (P \setminus P_n)$. ✓c ∎(Cor)

We may see the importance of $\mathcal{L}$-N uniformization in descriptive combinatorics and CBER theory. For now, we record one particularly useful fact.

Cor: Suppose that X, Y are std Borel and that $f : X \to Y$ is a ctbl-to-one Borel function. Then $f[X]$ is Borel.

pf: The set $\bar{f} = \{(f(x), x) : x \in X\} \subseteq Y \times X$ is Borel with countable vertical sections. Thus $\mathrm{Proj}_Y[\bar{f}] = f[X]$ is Borel by ⓐ above. ∎(Cor)





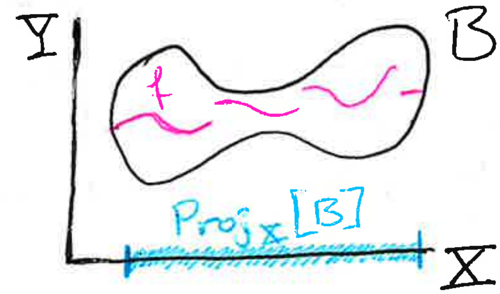
DST
Lecture 21

More on Uniformization

Our vague setup: X and Y are "nice" spaces.
$B \subseteq X \times Y$ is a "nice" set.

ⓐ How "nice" is $\mathrm{Proj}_X[B]$?

ⓑ Is there a "nice" uniformization?
I.e., $f: \mathrm{Proj}_X[B] \longrightarrow Y$
with $f \subseteq B$.



With a touch of revisionist history, we have seen a couple of instances of this setup:

Thm (Lusin): X, Y std Borel, $B \subseteq X \times Y$ Borel s.t. $\forall x \in X \ |B_x| \leq 1$. Then $\mathrm{Proj}_X[B]$ is Borel, and B is its own Borel uniformization.

Thm (Lusin-Novikov): X, Y std Borel, $B \subseteq X \times Y$ Borel s.t. $\forall x \in X \ |B_x| \leq \aleph_0$. Then $\mathrm{Proj}_X[B]$ is Borel and B has a Borel uniformization.

Today we uniformize analytic sets.

Def: For std Borel X, denote by $\sigma(\Sigma^1_1)$ the σ-algebra generated by the analytic subsets of X.

2 **Remarks:** (a) Every Borel set is $\sigma(\underset{\sim}{\Sigma}^1_1)$

(b) Given any compatible Polish topology τ on X, every $\sigma(\underset{\sim}{\Sigma}^1_1)$ set is τ-BP.

(c) Given any Borel probability measure μ on X, every $\sigma(\underset{\sim}{\Sigma}^1_1)$ set is μ-measurable.

Def: Given std Borel X, Y, a function $f: X \to Y$ is <u>$\sigma(\underset{\sim}{\Sigma}^1_1)$-measurable</u> if preimages of Borel sets are $\sigma(\underset{\sim}{\Sigma}^1_1)$.

Thm (Jankov, von Neumann): Suppose that X, Y are std Borel and that $A \subseteq X \times Y$ is analytic. Then (a) $P = \mathrm{Proj}_X[A]$ is analytic. ✓

(b) A admits a $\sigma(\underset{\sim}{\Sigma}^1_1)$-measurable uniformization $f: P \to Y$.

pf: We first handle a special case, and later argue that it isn't so special.

<u>Special case:</u> X is Polish, $Y = \omega^\omega$,

$A \subseteq X \times Y$ is closed, hence

$\forall x \in X \quad A_x$ is closed in ω^ω.

<u>Idea:</u> We know each A_x has the form $[T_x]$ for some tree $T_x \subseteq \omega^{<\omega}$. Our uniformization will follow the <u>leftmost branch</u> through T_x.

③ pf ($J \vee N$, special case, cont.)

More formally, we want for each $n \in \omega$ a $\sigma(\utilde{\Sigma}^1_1)$-measurable $f_n : P \to \omega^n$ s.t. $f(x)$ is the lex-least $s \in \omega^n$ with $A_x \cap \mathcal{N}_s \neq \emptyset$.

This implies $f_n(x) \sqsubseteq f_{n+1}(x)$, so $f = \lim_n f_n$ exists. Then f will be $\sigma(\utilde{\Sigma}^1_1)$-measurable, and moreover $f \subseteq A$ because A is closed.

We now officially build $f_n : P \to \omega^n$ by recursion

Base case: $f_0 : x \mapsto \emptyset$ $\quad x \in P \Rightarrow A_x \cap \mathcal{N}_\emptyset \neq \emptyset$

Recursive step: Given $f_n : P \to \omega^n$, $\forall k \in \omega$ we say $f_{n+1}(x) = f_n(x)^\frown k$ iff k is least with $A_x \cap \mathcal{N}_{f_n(x)^\frown k} \neq \emptyset$.

Claim: Each f_n is $\sigma(\utilde{\Sigma}^1_1)$-measurable.

pf(C): Suppose true for f_n, i.e., that $\forall s \in \omega^n$ the preimage $f_n^{-1}(\{s\})$ is $\sigma(\utilde{\Sigma}^1_1)$. We want to show $\forall s^\frown k \in \omega^{n+1}$ that $f_{n+1}^{-1}(\{s^\frown k\})$ is $\sigma(\utilde{\Sigma}^1_1)$.

Now $f_{n+1}(x) = s^\frown k$ iff:

- $\sigma(\utilde{\Sigma}^1_1)$ ▫ $x \in f_n^{-1}(\{s\})$, AND
- $\utilde{\Sigma}^1_1$ ▫ $x \in \mathrm{Proj}_{\mathcal{X}}[A \cap (\mathcal{X} \times \mathcal{N}_{s^\frown k})]$, AND
- $\utilde{\Pi}^1_1$ ▫ $\forall j < k \;\; x \notin \mathrm{Proj}_{\mathcal{X}}[A \cap (\mathcal{X} \times \mathcal{N}_{s^\frown j})]$

So $f_{n+1}^{-1}(\{s^\frown k\})$ is a Boolean combination of $\sigma(\utilde{\Sigma}^1_1)$ sets, thus is $\sigma(\utilde{\Sigma}^1_1)$ as desired. ■(C).

As discussed above, $\lim_n f_n$ is our uniformization.

■($J \vee N$, special case)

④ pf (J v N, cont.)

General case: Assume $A \neq \emptyset$. Fix Polish topologies on $\mathbb{X}, \mathbb{Y}$ and a continuous surj $\varphi : \omega^\omega \twoheadrightarrow A$ witnessing analyticity.

Define $B \subseteq \mathbb{X} \times \omega^\omega$ by

$$B = \{(x, z) : \mathrm{Proj}_{\mathbb{X}} \circ \varphi(z) = x\}$$

Now B is closed by continuity of $\mathrm{Proj}_{\mathbb{X}} \circ \varphi$.

Note: $x \in \mathrm{Proj}_{\mathbb{X}}[B]$ iff $\exists z \in \omega^\omega \, \exists y \in \mathbb{Y} \; \varphi(z) = (x, y)$
iff $\exists y \in \mathbb{Y} \; (x, y) \in A$
iff $x \in \mathrm{Proj}_{\mathbb{X}}[A]$.

By our special case, we have a $\sigma(\underset{\sim}{\Sigma}^1_1)$-meas uniformization $g : P \to \omega^\omega$ with $g \subseteq B$. Finally, $f = \mathrm{Proj}_{\mathbb{Y}} \circ \varphi \circ g$ is our desired $\sigma(\underset{\sim}{\Sigma}^1_1)$-meas uniformization of A. ■ (J v N)

Remarks: Suppose that $B \subseteq \mathbb{X} \times \omega^\omega$ is closed.

- The "complexity bump" from Borel to $\sigma(\underset{\sim}{\Sigma}^1_1)$ occurs when testing whether sets of the form $B_x \cap \mathcal{N}_s \subseteq \omega^\omega$ are empty.
- This complexity can be lowered back down to Borel in various situations:
  - ⓐ If each B_x is compact, Kőnig's Lemma reduces the complexity of searching for branches.
  - ⓑ If each B_x is non-meager / non-null, instead of testing whether $B_x \cap \mathcal{N}_s = \emptyset$, test whether $B_x \cap \mathcal{N}_s$ is meager/null.

DST

Lecture 22

Edge colorings of graphs

Def: Suppose that $G \subseteq X^2$ is a graph on X. An edge (B-)coloring of G is a function

$c: G \to B$ satisfying:

- $c(x,y) = c(y,x)$
- $c(x,y) = c(x,y') \Rightarrow y = y'$

Ex: G_0 on 2^ω admits a Borel edge ω-coloring

$$c: G_0 \to \omega \quad \text{via} \quad \left.\begin{array}{l}(s_n{}^\frown 0^\frown z, s_n{}^\frown 1^\frown z) \\ (s_n{}^\frown 1^\frown z, s_n{}^\frown 0^\frown z)\end{array}\right\} \mapsto n.$$

This is not special.

Thm (ess. Feldman-Moore, 1977):

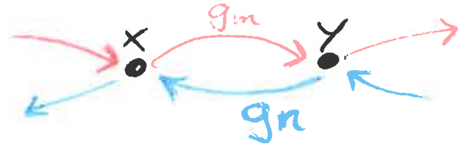
Suppose that G is a locally countable Borel graph on standard Borel X. Then G admits a Borel edge ω-coloring.

pf: By Lusin-Novikov, we may find Borel partial functions $f_n: X \rightharpoonup X$ with $G = \bigcup_n f_n$. We might as well disjointify them, putting $g_n = f_n \setminus \bigcup_{m<n} f_m$ to get Borel $g_n: X \rightharpoonup X$ such that $G = \bigsqcup_n g_n$.

② <u>pf (Thm, cont.)</u>

<u>Hope</u>: put $c(x,y) = n$ iff $g_n : x \mapsto y$.

<u>Problem</u>: $c(x,y) \overset{?}{=} c(y,x)$



We need to "induce symmetry" somehow.

The theorem is trivial when $\mathbb{X}$ is countable. By the Borel isomorphism theorem, we thus lose no generality by assuming $\mathbb{X} = 2^\omega$.

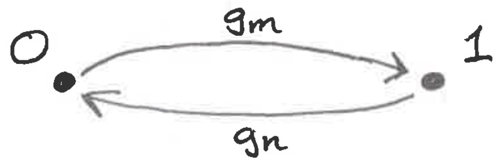
Given $x \neq y$ in 2^ω, put

$$\delta(x,y) = \text{least } k \text{ with } x(k) \neq y(k).$$

So $\delta : G \to \omega$ is Borel and $\delta(x,y) = \delta(y,x)$.

Finally, define $c : G \to \omega^3$ by

$$c(x,y) = (k,m,n) \text{ iff } \delta(x,y) = k \text{ AND}$$
$$\left(\begin{array}{ll} & x(k)=0 \text{ and } g_m : x \mapsto y \text{ and } g_n : y \mapsto x \\ \text{OR} & x(k)=1 \text{ and } g_m : y \mapsto x \text{ and } g_n : x \mapsto y \end{array}\right)$$



<u>Claim 0</u>: c is Borel.

<u>pf (C0)</u>: Each $c^{-1}(\{k,m,n\})$ is a Boolean combination of Borel sets. ■ (C0)

<u>Claim 1</u>: $c(x,y) = c(y,x)$.

<u>pf (C1)</u>: Inspect the above picture. ■ (C1)

③ <u>pf (Thm, cont.)</u>

<u>Claim 2</u>: If $c(x,y) = c(x,y')$, then $y = y'$.

<u>pf (C2)</u>: Say $c(x,y) = c(x,y') = (k,m,n)$. Then in particular, $\delta(x,y) = \delta(x,y') = k$.

<u>Case 0</u>: $x(k) = 0$. Then $g_m : x \mapsto y, \; x \mapsto y'$, so $y = y'$. ✓

<u>Case 1</u>: $x(k) = 1$. Then $g_n : x \mapsto y, \; x \mapsto y'$, so $y = y'$. ✓

■(C2)

This means that c is a Borel edge ω-coloring. ■(Thm)

<u>Rmk</u>: The "true" Feldman-Moore theorem is a dynamical analog of this combinatorial result. To discuss it, we need a few more definitions.

<u>Def</u>: ⓐ Given a topological group Γ and std Borel space X, an action $\Gamma \curvearrowright X$ is <u>Borel</u> if the function $(\gamma, x) \longmapsto \gamma \cdot x$ is Borel.

ⓑ If Γ is countable (with the discrete topology), this is tantamount to requiring for all $\gamma \in \Gamma$ that the map $x \longmapsto \gamma \cdot x$ is Borel.

<u>Def</u>: Given std Borel X and $E \subseteq X^2$, we say that E is a <u>countable Borel equivalence relation</u> (CBER) on X if it is a Borel eq. rel. s.t. $\forall x \in X$ the equiv class $[x]_E$ is countable.

Thm (Feldman-Moore, 1977): Suppose that E is a CBER on std Borel X. Then there is a countable group Γ and a Borel action $\Gamma \curvearrowright X$ whose orbit eq. rel. is E. I.e., $x \, E \, y$ iff $\exists \gamma \in \Gamma$ $\gamma \cdot x = y$.

pf: Fix a Borel edge ω-coloring $c : E \setminus \Delta \to \omega$. For each $n \in \omega$, put $A_n = \mathrm{Proj}\,[c^{-1}(\{n\})]$, so A_n is the Borel set of vertices with a (unique!) incident edge of color n. Define for each $n \in \omega$ a Borel involution

$$\imath_n : x \longmapsto \begin{cases} y & \text{if } x \in A_n \text{ and } c(x,y) = n \\ x & \text{if } x \notin A_n. \end{cases}$$

Let Γ be the (countable) group of Borel automorphisms generated by these involutions, and let Γ act on X by evaluation. ■ (Thm)

Remark: This actually builds an action of $\Gamma = \mathop{\ast}\limits_{n \in \omega} (\mathbb{Z}/2\mathbb{Z})$, the free product of countably many involutions. This is not so impressive, as every countable group arises as a quotient of a subgroup of this Γ.

DST

Lecture 23

Smooth equivalence relations

Def: An equivalence relation E on std Borel X is called smooth (or concretely classifiable, or tame, or ...) if it is Borel reducible to equality on some std Borel Y. In symbols:

$$E \leq_B \Delta_Y .$$

Unfolded, this means $\exists$ Borel $f : X \to Y$ s.t. $x \, E \, x'$ iff $f(x) = f(x')$.

Examples:

⓪ Equality. ☺

① The orbit equivalence relation of $\mathbb{Z} \curvearrowright \mathbb{R}$. One such reduction is $x \longmapsto x - \lfloor x \rfloor$.

② $G = GL_n(\mathbb{C})$, the group of invertible $(n \times n)$-matrices with complex entries. Consider the conjugation action $G \curvearrowright G$ via $A \cdot B = ABA^{-1}$. Its orbit eq. rel. is smooth by Jordan normal form.

③ $X = \{x \in 2^{\omega \times \omega} : x$ codes a dense lin. order on $\omega\}$. $S_\infty \curvearrowright X$ via the logic action

$$\sigma \cdot x : (i,j) \mapsto x(\sigma^{-1}(i), \sigma^{-1}(j)).$$

The orbit eq. rel is smooth as there are exactly four (Borel) orbits.

② Non-examples:

① $\mathbb{E}_0$ on 2^ω

② Orbit eq. rel of $\mathbb{Q} \curvearrowright \mathbb{R}$.

Here are two notions closely related to smoothness.

Def: Suppose that E is an equiv. relation on X.

ⓐ A *transversal* of E is a subset of X meeting each E-class in exactly one point.

ⓑ A *selector* for E is a function $f: X \to X$ such that
- $f(x) \mathrel{E} x$
- $x \mathrel{E} x' \Rightarrow f(x) = f(x')$.

Recall: A CBER is a Borel eq. rel on a std Borel space with all classes countable.

Thm: Suppose that E is a CBER on std Borel X.

TFAE: $\boxed{\mathrm{I}}$ E is smooth

$\boxed{\mathrm{II}}$ E admits a Borel transversal

$\boxed{\mathrm{III}}$ $\chi_B(E \setminus \Delta_X) \leq \aleph_0$

$\boxed{\mathrm{IV}}$ E admits a Borel selector.

Remark: This theorem fails miserably for Borel equivalence relations with uncountable classes. Notably, $\boxed{\mathrm{III}}$ never holds...

pf (Thm): By Feldman-Moore, fix a countable group Γ and a Borel action $\Gamma \curvearrowright X$ whose orbit equivalence relation is E.

$\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$: Fix a Borel reduction $f: X \to Y$ from E to Δ_Y. Consider the now-familiar $\mathcal{F} = \{(f(x), x) : x \in X\} \subseteq Y \times X$. Its non-empty vertical sections are E-classes, so in particular are countable. Lusin-Novikov grants a Borel uniformization $g: P \to X$ of $\mathcal{F}$. This g is injective, and $g[P]$ is a Borel transversal of E as desired. ✓

Remark: We actually just showed that every countable-to-one Borel function admits a Borel right inverse.

$\boxed{\text{II}} \Rightarrow \boxed{\text{III}}$: Fix a Borel transversal $B \subseteq X$ of E. For each $\gamma \in \Gamma$, $\gamma \cdot B$ is still a (Borel) transversal of E, hence is $(E \setminus \Delta_X)$-indep. As $X = \Gamma \cdot B = \bigcup_{\gamma \in \Gamma} \gamma \cdot B$, we have covered X by countably many Borel $(E \setminus \Delta_X)$-indep sets, establishing $\chi_B(E \setminus \Delta_X) \leq \aleph_0$. ✓

pf(Thm, cont.)

$\boxed{\text{III}} \Rightarrow \boxed{\text{IV}}$: Fix a Borel coloring $c: X \to \omega$ of $E \setminus \Delta_X$. We build a selector $f: X \to X$ by sending x to the element of $[x]_E$ that receives the smallest color among elements of $[x]_E$. More formally,

$$f(x) = y \text{ iff } x \mathrel{E} y \wedge \forall \gamma \in \Gamma \; c(y) \leq c(\gamma \cdot x).$$

This selector f is manifestly Borel. ✓

Remark: This trick of using Lusin-Novikov (or Feldman-Moore) to quantify over an E-class using "natural number quantifiers" is very handy in studying CBERs.

$\boxed{\text{IV}} \Rightarrow \boxed{\text{I}}$: Just check that a Borel selector is also a Borel reduction from E to Δ_X. ✓

∎ (Thm)

Later, we will see a dual theorem:

Thm: Suppose that E is a CBER on std Borel X.

TFAE: $\boxed{\text{I}}$ $\mathbb{E}_0 \leq_B E$

$\boxed{\text{II}}$ $\chi_B(E \setminus \Delta_X) > \aleph_0$.

Cor (Glimm-Effros dichotomy for CBERs):

Suppose that E is a CBER. Exactly one:

$\boxed{\text{I}}$ $E \leq_B \Delta_{\mathbb{R}}$

$\boxed{\text{II}}$ $\mathbb{E}_0 \leq_B E$.

DST

Lecture 24

Descriptive combinatorics of locally finite graphs

Recently, we have seen many applications of some graph-theoretic ideas in descriptive set theory. E.g.:

Thm (Kechris-Solecki-Todorcevic, 1999):
Suppose that G is an analytic graph on a Hausdorff space X. Exactly one:
$\boxed{\text{I}}$ $\chi_B(G) \leq \aleph_0$
$\boxed{\text{II}}$ There is a continuous hom from $\mathbb{G}_0$ to G.

Thm (ess. Feldman-Moore, 1977):
Suppose that G is a locally countable Borel graph on std Borel X. Then G admits a Borel edge ω-coloring, i.e., there are Borel partial involutions $\iota_n : X \to X$ with $G = \bigcup_n \iota_n$.

Cor: Suppose that G is a loc ctbl Borel graph on std Borel X and that $A \subseteq X$ is Borel. Then its set of G-neighbors

$$N_G(A) = \{x \in X : \exists a \in A \;\; x \, G \, a\}$$

is also Borel.

pf (Cor): $N_G(A) = \bigcup_n \iota_n[A] = \bigcup_n \iota_n^{-1}(A)$. ■ (Cor)

② Today, we explore locally finite graphs.

<u>Def</u>: A graph G on X is <u>locally finite</u> if $\forall x \in X$ $G_x = \{y \in X : x \, G \, y\}$ is finite.

<u>Prop A</u> (KST): Suppose that G is a loc fin Borel graph on std Borel X. Then $\chi_B(G) \leq \aleph_0$.

<u>pf</u>: Enumerate a countable algebra $\mathcal{A} = \{A_n : n \in \omega\}$ of Borel sets generating the Borel σ-alg on X. Consider $B_n = A_n \setminus N_G(A_n)$, a Borel G-indep set.

<u>Claim</u>: $\{B_n : n \in \omega\}$ covers X.

<u>pf(C)</u>: Suppose $x \in X$ is arbitrary. For each $y \in G_x$ find $A_y \in \mathcal{A}$ with $x \in A_y$ and $y \notin A_y$. Put $A = \bigcap \{A_y : y \in G_x\}$, so $A \in \mathcal{A}$. Then $x \in A$ and $G \cap A = \emptyset$, and hence $x \in A \setminus N_G(A)$. ■(C)

We have shown $\chi_B(G) \leq \aleph_0$. ■(Prop A)

<u>Prop B</u> (KST): Suppose that G is a loc fin Borel graph on std Borel X, and that $A \subseteq X$ is a Borel G-indep. set. Then there is a Borel maximal G-indep set $A^* \supseteq A$.

<u>pf</u>: Let $\{B_n : n \in \omega\}$ enumerate a cover of X by Borel G-indep sets. Recursively define:

$$A_0 = A$$
$$A_{n+1} = A_n \cup (B_n \setminus N_G(A_n)).$$

Then $A^* = \bigcup_n A_n$ works. ■(Prop B)

3 **Def:** Given $d \in \omega$, we say that a graph G on X has degree bounded by d if $\forall x \in X \; |G_x| \leq d$.

Prop C (KST): Suppose that G is a Borel graph on std Borel X with degree bounded by d. Then $\chi_B(G) \leq d+1$.

pf: Induct on $d \in \omega$:

$d=0$:

$d>0$: Fix a Borel max'l G-indep set $A \subseteq X$. Then $G \upharpoonright (X \setminus A)$ has degree bounded by $d-1$, hence $\chi_B(G \upharpoonright (X \setminus A)) \leq d$. Use A as the last color. ■ (Prop C).

Examples: Bernoulli shifts

Suppose that Γ is a countably infinite group. Consider $2^\Gamma = \{x : \Gamma \to 2\}$ equipped with product topology and let μ be the $(\frac{1}{2}, \frac{1}{2})$-coin-flip meas. Then $\Gamma \curvearrowright 2^\Gamma$ via shifts: $\gamma \cdot x : \delta \mapsto x(\gamma^{-1}\delta)$.

OR $(\gamma \cdot x)(\gamma \cdot \delta) = x(\delta)$.

Let X be the free part of this action,

$$X = \{x \in 2^\Gamma : \forall \gamma \neq e \;\; \gamma \cdot x \neq x\}$$

[HW] X is a dense G_δ in 2^Γ. Also μ-conull.

Finally, for symmetric $S \subseteq \Gamma \setminus \{e\}$ define a Borel graph $G = G_{\Gamma, S}$ on X by

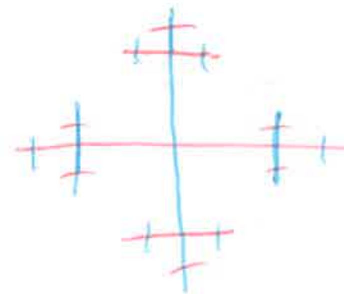
$$x \; G \; y \text{ iff } \exists \gamma \in S \;\; \gamma \cdot x = y.$$

Example 1: $\Gamma = \mathbb{Z}$, $S = \{\pm 1\}$

Then G has degree bounded by 2. Prop C says $\chi_B(G) \leq 3$. Arguments similar to irrat'l rotation show $\chi_{BP}(G) = \chi_\mu(G) = \chi_B(G) = 3$.

Example 2: $\Gamma = F_2 = \langle a, b \rangle$

$S = \{a, b\}^{\pm}$



So $\chi_B(G) \leq 5$

- (Marks, 2016): $\chi_B(G) = 5$
- (-Miller, 2016): $\chi_{BP}(G) = 3$
- (-Marks-Tucker-Drob, 2016): $\chi_\mu(G) \in \{3, 4\}$

Example n: $\Gamma = F_n$, $S =$ free gen set.

- (Marks, 2016): $\chi_B(G_n) = 2n+1$
- (-Miller, 2016): $\chi_{BP}(G_n) = 3$
- (-Kechris, 2013): $\chi_\mu(G_n) \to \infty$

  (Bernshteyn, 2019): $\chi_\mu(G_n) \approx \frac{n}{\log(n)}$.

DST

Lecture 25

Today's goal:

Thm: Suppose that E is a CBER on std Borel X.

TFAE: $\boxed{\text{I}}$ $\mathbb{E}_0 \leq_B E$

$\boxed{\text{II}}$ $\chi_B(E \setminus \Delta_X) > \aleph_0$.

We first collect a lemma.

Lemma: Suppose that F is an eq rel on 2^ω satisfying

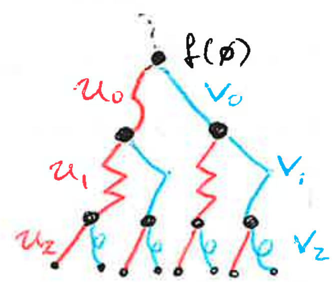
- $\mathbb{E}_0 \subseteq F$
- F is meager (in $2^\omega \times 2^\omega$).

Then $\mathbb{E}_0 \leq_B F$.

pf($\mathcal{L}$, sketch): We tweak Mycielski. Fix a decreasing sequence $V_n \subseteq 2^\omega \times 2^\omega$ of open dense sets with $\bigcap_n V_n$ disjoint from F.

Def: A function $f : 2^{<\omega} \to 2^{<\omega}$ is an aligned embedding if there are $u_n, v_n \in 2^{<\omega}$ s.t.

- $0 \sqsubseteq u_n$, $1 \sqsubseteq v_n$
- $\text{len}(u_n) = \text{len}(v_n)$
- $\forall s \in 2^n$ $f : \begin{array}{l} s^\frown 0 \longmapsto f(s)^\frown u_n \\ s^\frown 1 \longmapsto f(s)^\frown v_n \end{array}$



So $f : 0110\ldots \longmapsto f(\emptyset)^\frown u_0{}^\frown v_1{}^\frown v_2{}^\frown u_3{}^\frown \cdots$

Ex: There is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ s.t. $\forall s, t \in 2^n$ $\mathcal{N}_{f(s^\frown 0)} \times \mathcal{N}_{f(t^\frown 1)} \subseteq V_n$.

② pf ($\mathcal{L}$, cont.): Fix such an aligned embedding f, and consider $\varphi = \lim f : 2^\omega \to 2^\omega$.

Claim 0: $\sigma \mathbb{E}_0 \tau \Rightarrow \varphi(\sigma) \mathbb{E}_0 \varphi(\tau)$ [hence $\varphi(\sigma) F \varphi(\tau)$].

pf (C0): Suppose $\begin{array}{l}\sigma = s^\frown\rho \\ \tau = t^\frown\rho\end{array}$, so $\begin{array}{l}\varphi(\sigma) = f(s)^\frown\rho' \\ \varphi(\tau) = f(t)^\frown\rho'\end{array}$. $\blacksquare$(C0)

Claim 1: $\sigma \not\mathbb{E}_0 \tau \Rightarrow \varphi(\sigma) \not F \varphi(\tau)$.

pf (C1): To show $\varphi(\sigma) \not F \varphi(\tau)$, it suffices to show ~~that~~ $(\varphi(\sigma), \varphi(\tau)) \in \bigcap_n V_n$. As $(V_n)_{n\in\omega}$ is decreasing, it's enough to check that

$$\forall m \ \exists n \geq m \ (\varphi(\sigma), \varphi(\tau)) \in V_n$$

Assume $\sigma \not\mathbb{E}_0 \tau$, so $\exists n \geq m$ with $\sigma(n) \neq \tau(n)$.

Say $\begin{array}{l}s^\frown 0 \sqsubseteq \sigma \\ t^\frown 1 \sqsubseteq \tau\end{array}$ and thus $\begin{array}{l}\varphi(\sigma) \in \mathcal{N}_{f(s^\frown 0)} \\ \varphi(\tau) \in \mathcal{N}_{f(t^\frown 1)}.\end{array}$

$\star$ then implies $(\varphi(\sigma), \varphi(\tau)) \in V_n$ as desired. $\blacksquare$(C1)

The two claims show that φ is a continuous (hence Borel) reduction from $\mathbb{E}_0$ to F. $\blacksquare$($\mathcal{L}$)

pf (Thm): $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$: Suppose that $\mathbb{E}_0 \leq_B E$ and, towards a contradiction, that $\chi_B(E \setminus \Delta_X) \leq \aleph_0$. As discussed in Lecture 23, this implies that $E \leq_B \Delta_{\mathbb{R}}$. Composing the reductions would yield $\mathbb{E}_0 \leq_B \Delta_{\mathbb{R}}$, which is absurd. ✓

③ <u>pf (Thm, cont.)</u>

$\boxed{\mathrm{II}} \Rightarrow \boxed{\mathrm{I}}$: Suppose that $\chi_B(E \setminus \Delta_X) > \aleph_0$. Then the G_0 dichotomy yields a Borel homomorphism $\Theta : 2^\omega \to X$ from $\mathbb{G}_0$ to $E \setminus \Delta_X$. Define F on 2^ω as the pull-back of E via Θ, so

$$\sigma \, F \, \tau \text{ iff } \Theta(\sigma) \, E \, \Theta(\tau).$$

Certainly, F is a Borel eq. rel. on 2^ω, and moreover Θ witnesses that $F \leq_B E$.

<u>Claim 0</u>: $\mathbb{E}_0 \subseteq F$.

<u>pf (C0)</u>: $\mathbb{G}_0 \subseteq F$. ■(C0)

<u>Claim 1</u>: F is meager (in $2^\omega \times 2^\omega$).

<u>pf (C1)</u>: As usual, Kuratowski-Ulam reduces this to checking that each F-class is meager. For $\sigma \in 2^\omega$ we compute

$$\begin{aligned} [\sigma]_F &= \Theta^{-1}([\Theta(\sigma)]_E) \\ &= \Theta^{-1}(\{x \in X : x \, E \, \Theta(\sigma)\}) \\ &= \bigcup \{\Theta^{-1}(\{x\}) : x \, E \, \Theta(\sigma)\}. \end{aligned}$$

As $[\Theta(\sigma)]_E$ is countable, we have written $[\sigma]_F$ as a countable union of $\mathbb{G}_0$-indep Borel sets. Thus, $[\sigma]_F$ is meager. ■(C1).

So our lemma applies, granting a Borel reduction φ from $\mathbb{E}_0$ to F. Then $\Theta \circ \varphi$ is a Borel reduction from $\mathbb{E}_0$ to E. ✓ ■(Thm)

④ A map of CBERs so far: $\xrightarrow{\leq_B}$

$$\underbrace{\Delta_0 \quad \Delta_1 \quad \cdots \quad \Delta_{\mathbb{N}} \quad \Delta_{\mathbb{R}}}_{\text{smooth}} \quad \mathbb{E}_0$$

It would be nice to have an "intrinsic characterization" of when a CBER is bireducible with $\mathbb{E}_0$.

Def: ⓐ A CBER is a Borel eq. rel. on a std Borel space with all classes countable.

ⓑ A CBER is aperiodic if every class is infinite.

ⓒ An FBER is a CBER with all classes finite.

Remarks:

ⓐ Given any CBER, there is a Borel partition of its underlying space into invariant pieces: an aperiodic part and a finite part.

ⓑ Any FBER is smooth.

Def: A CBER E on X is hyperfinite if there is a sequence $(F_n)_{n\in\omega}$ of FBERs on X s.t.
- $F_n \subseteq F_{n+1}$
- $\bigcup_n F_n$.

"E is an increasing union of FBERs."

Next goal: A CBER E is hyperfinite iff $E \leq_B \mathbb{E}_0$.

Ex: $E \leq_B \mathbb{E}_0 \Rightarrow E$ is hyperfinite (for CBER E)

DST

Lecture 26

Hyperfiniteness criteria (part 1)

Recall: A CBER is hyperfinite if it is an increasing union of FBERs.

Prop: Suppose that E, F are CBERs on X, Y respectively, and that $\varphi : X \to Y$ is a ctbl-to-one Borel hom from E to F. If F is hyperfinite, then E is also hyperfinite.

pf: Fix FBERs F_n on Y witnessing hfness of F. By Lusin-Novikov, find Borel $X_n \subseteq X$ with $X = \bigcup_n X_n$ s.t. $\forall n \in \omega$ $\varphi \upharpoonright X_n$ is injective. We now define E_n on X by

$x \, E_n \, x'$ iff $x = x'$ OR

$$[x \, E \, x' \text{ AND } x, x' \in \bigcup_{m<n} X_m \text{ AND } \varphi(x) \, F_n \, \varphi(x')].$$

It is routine to check that $(E_n)_{n\in\omega}$ is an increasing sequence of FBERs with $E = \bigcup_n E_n$.

■ (Prop)

Remark: This proposition applies in two important settings: $E \leq_B F$ and $E \subseteq F$.

② **Def:** $\mathbb{E}_0^\omega$ is the CBER of eventual equality on ω^ω.

So $x \,\mathbb{E}_0^\omega\, y$ iff $\exists m\ \forall n \geq m\ \ x(n) = y(n)$.

Warm-up: $\mathbb{E}_0^\omega \leq_B \mathbb{E}_0$.

pf: $x \longmapsto \mathcal{X}_x = \begin{array}{|cccc} 0&0&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&1&0&1\ \cdots \\ \hline \end{array} \in 2^{\omega\times\omega} \cong 2^\omega$. ∎ (w-u)

Thm (Dougherty-Jackson-Kechris, 1994):

Suppose that E is a CBER on X. TFAE:

$\boxed{\mathrm{I}}$ E is hyperfinite

$\boxed{\mathrm{II}}$ $E \leq_B \mathbb{E}_0$.

pf: $\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$: Fix an increasing sequence $(F_n)_{n\in\omega}$ of FBERs witnessing hfness of E. Without loss, we assume $X = 2^\omega$. Let $<$ denote the lexicographical order on 2^ω.

Def: For $x \in 2^\omega$ and $n \in \omega$, let the <u>n-brick</u> of x denote the following array of 0s and 1s: lex-order $[x]_{F_n}$ as $x_0 < x_1 < \cdots < x_{k-1}$ and record the first n bits of each x_i.

E.g.:

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ \hline x_0 & x_1 & x_2 & x_3 & x_4 \end{array} \quad \text{3-brick}$$

$$\underbrace{x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4}_{[x]_{F_3}}$$

③ pf (Thm, cont.)

Let $B_n(x)$ denote the n-brick of x.

Facts: ▫ $x \longmapsto B_n(x)$ is Borel

▫ $x \, F_n \, y \Rightarrow B_n(x) = B_n(y)$.

The idea is that $B_n(x)$ records a finite approximation of x's F_n-class. We will also need to record how x's F_n-class sits inside of its F_{n+1}-class.

Def: For $x \in 2^\omega$ and $n \in \omega$, let $r_n(x) \in \omega^{<\omega}$ record the indices that $[x]_{F_n}$ occupies in $[x]_{F_{n+1}}$ with everything lex-ordered.

E.g., if $[x]_{F_2}$ and $[x]_{F_3}$ are

$$\underbrace{\begin{array}{ccc} \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array}}_{[x]_{F_2}} \qquad\qquad \underbrace{\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}}_{[x]_{F_3}}$$

then $r_2(x) = 1, 2, 4$.

More facts: ▫ $x \longmapsto r_n(x)$ is Borel

▫ $x \, F_n \, y \Rightarrow r_n(x) = r_n(y)$.

(4) pf (Thm, cont.)

Claim: The map $\varphi : x \mapsto (B_n(x), r_n(x))_{n\in\omega}$ is a Borel reduction from E to $\ddot{\mathbb{E}}_0^{\omega}$.

pf (C): First, suppose $x \, E \, y$ and fix $m \in \omega$ with $x \, F_m \, y$. Then for all $n \geq m$ we have $B_n(x) = B_n(y)$ and $r_n(x) = r_n(y)$. In other words, $\varphi(x) \, \ddot{\mathbb{E}}_0^{\omega} \, \varphi(y)$.

Conversely, suppose that $\varphi(x) \, \ddot{\mathbb{E}}_0^{\omega} \, \varphi(y)$ and fix $m \in \omega$ s.t. $\forall n \geq m$ $B_n(x) = B_n(y)$ and $r_n(x) = r_n(y)$.
Lex order $[x]_{F_m}$ as $x_0 < x_1 < \cdots < x_{k-1}$, ~~and~~
$[y]_{F_m}$ as $y_0 < y_1 < \cdots < y_{k-1}$.

Fix $i < k$ with $y = y_i$. Observe that $x_i = y_i$ as their tails are determined by $B_n(x)$ and $r_n(x)$.
In particular, $y \in [x]_{F_m}$ and thus $x \, E \, y$. ■(C)

So $E \leq_B \ddot{\mathbb{E}}_0^{\omega}$, hence $E \leq_B \mathbb{E}_0$ by warm-up. ✓

$\boxed{\text{II}} \Rightarrow \boxed{\text{I}}$: We know that $\mathbb{E}_0$ is hyperfinite.
Then if $E \leq_B \mathbb{E}_0$, our proposition implies that E is hyperfinite as well. ✓

■(Thm)

DST

Lecture 27

Hyperfiniteness criteria (part 2)

Today we discuss algebraic aspects of hyperfiniteness.

Def: Given an eq. rel. E on X, a sequence $(A_n)_{n\in\omega}$ is called a vanishing marker sequence (for E) if

- $A_{n+1} \subseteq A_n$
- $\bigcap_n A_n = \emptyset$
- $[A_n]_E = X$

Remark [AC]: Such a sequence exists exactly when every E-class is infinite.

Marker Lemma: Suppose that E is an aperiodic CBER on X. Then E admits a vanishing Borel marker sequence.

pf: WLOG $X = 2^\omega$. For each $n \in \omega$ we define $f_n : 2^\omega \to 2^n$ via $x \longmapsto$ lex least $s \in 2^n$ s.t. $[x]_E \cap N_s$ is infinite. Each f_n is Borel and E-invariant. Define Borel sets $B_n = \{x \in 2^\omega : x \in N_{f_n(x)}\}$, so for all $x \in 2^\omega$ the set $[x]_E \cap B_n = [x]_E \cap N_{f_n(x)}$ is infinite.

Claim: $B_\omega = \bigcap_n B_n$ meets each E-class at most once.

pf(C): Suppose that $x \, E \, y$ and both are in B_ω. For all n, $f_n(x) = f_n(y)$, and thus x and y share all initial segments. This means $x = y$. ■(C).

Now, $A_n = B_n \setminus B_\omega$ works as our marker sequence.

■(M.L.)

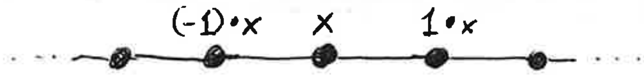
Thm (Slaman–Steel, 1988):

Suppose that X is a CBER on X. TFAE:

$\boxed{\text{I}}$ E is the orbit eq. rel. of a Borel action $\mathbb{Z} \curvearrowright X$

$\boxed{\text{II}}$ E is hyperfinite.

pf: $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$. FBERs are trivially hyperfinite, so WLOG E is aperiodic. Each E-class is



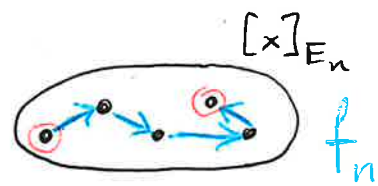
Fix a vanishing Borel marker sequence $(A_n)_{n\in\omega}$.
Put $X_{min} = \{x \in X : \exists n \in \omega \ \inf\{k \in \mathbb{Z} : k\cdot x \in A_n\} \neq -\infty\}$.
"There is a leftmost element of A_n in $[x]_E$."
Then X_{min} is Borel and E-invariant, and moreover $E \restriction X_{min}$ is smooth as the leftmost elements of A_n form a Borel transversal. Analogously define and analyze X_{max}. As smooth CBERs are hyperfinite, WLOG we may assume X_{min} and X_{max} are empty, i.e., that each A_n meets every E-class coinitially and cofinally.

Now define FBERs E_n on X by $x \, E_n \, y$ iff $x = y$ or they are connected after deleting A_n.
I.e., if $\{x, 1\cdot x, 2\cdot x, \ldots, y\} \cap A_n = \emptyset$.
As shown by the travel mug, these FBERs witness the hyperfiniteness of E. ✓

③ pf (Thm, cont.)

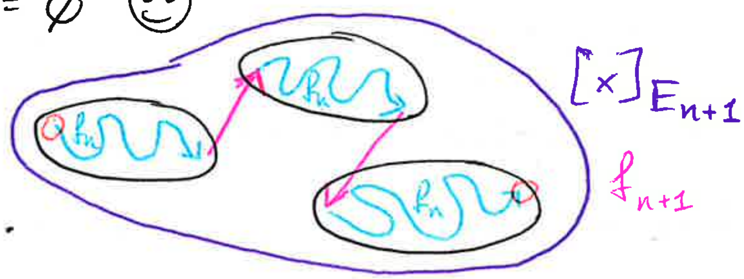
$\boxed{\text{II}} \Rightarrow \boxed{\text{I}}$: We run the travel mug argument in reverse. Fix FBERs E_n on X witnessing hfness of E, and WLOG $E_0 = \Delta_X$. We recursively construct Borel partial injections $f_n : X \to X$ satisfying $\forall x \in X$:

- the f_n-orbit of x is $[x]_{E_n}$
- $|[x]_{E_n} \setminus \mathrm{dom}(f_n)| = |[x]_{E_n} \setminus \mathrm{im}(f_n)| = 1$
- $f_n \subseteq f_{n+1}$



Stage 0: $f_0 = \emptyset$ ☺

Stage n+1:

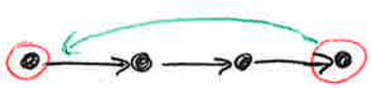


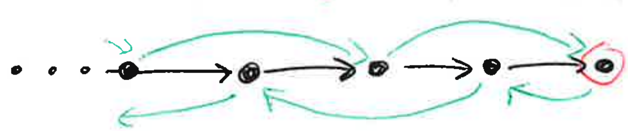
Put $f = \bigcup_n f_n$.

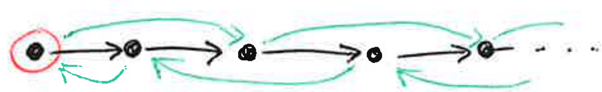
This f is a Borel partial injection satisfying

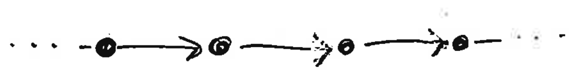
- the f-orbit of x is $[x]_E$.
- $|[x]_E \setminus \mathrm{dom}(f)| \leq 1$ and $|[x]_E \setminus \mathrm{im}(f)| \leq 1$.

Each case of $(|[x]_E \setminus \mathrm{dom}(f)|, |[x]_E \setminus \mathrm{im}(f)|)$ occurs on an E-invariant Borel set and is easy to handle:

(1, 1) 

(1, 0) 

(0, 1) 

(0, 0)  [already a $\mathbb{Z}$-action!] ✓

■ (Thm)

④ This leads to a natural open question:

Q: Which countable groups are "hyperfinite," i.e., have the property that all of their Borel actions yield hyperfinite orbit eq. rels?

Some known examples:

- $\mathbb{Z}$ (Slaman-Steel, 1988)
- Fin gen abelian (Weiss, 1984)
- Fin gen nilpotent (Jackson-Kechris-Louveau, 2002)
- Abelian (Gao-Jackson, 2015)
- Nilpotent (Schneider-Seward, 2024)
- Polycyclic + (-Jackson-Marks-Seward-TuckerDrob, 2023)

The question is still open for some important classes of groups. For example:

- Solvable
- Subexponential growth
- Amenable.

Next time, we will see that there is no hope of pushing this further: every non-amenable group has a Borel action with NON-hyperfinite orbit eq. rel.

DST

Lecture 28

Amenability and hyperfiniteness

We fix a standard Borel space X.

Def: A Borel probability measure on X is a function $\mu : \mathcal{B}(X) \to [0,1]$ satisfying

- $\mu(X) = 1$
- $\forall$ pairwise disjoint Borel $(A_n)_{n \in \omega}$, $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$.

We now fix such a Borel prob meas μ on X.

Def: Suppose that Γ is a countable group acting in a Borel fashion on X. We say that the action is μ-preserving (or that μ is Γ-invariant) if for all Borel $A \subseteq X$ and $\gamma \in \Gamma$, $\mu(\gamma \cdot A) = \mu(A)$.

Example: The Bernoulli shift action $\Gamma \curvearrowright 2^\Gamma$ preserves the $(\frac{1}{2}, \frac{1}{2})$-product-measure $\mu_{1/2}$ on 2^Γ.

Remark: When Γ is countably infinite, the free part of the action $\Gamma \curvearrowright 2^\Gamma$ has $\mu_{1/2}$-measure 1.

Def: Suppose that C is a non-$\emptyset$ finite set. For any set A, we define

$$m_C(A) = \frac{|A \cap C|}{|C|} \in [0,1].$$

(2) <u>Lemma:</u> Suppose that $\Gamma \curvearrowright X$ in a μ-pres Borel fashion with orbit eq. rel. E. Suppose also that $F \subseteq E$ is an FBER. Then for Borel $A \subseteq X$

$$\mu(A) = \int m_{[x]_F}(A)\, d\mu(x).$$

<u>pf($\mathcal{L}$):</u> WLOG, we may assume that F is an nBER, i.e., that every F-class has n elements. This is because there is a partition $X = \bigsqcup_n X_n$ into F-invariant Borel sets with $F \restriction X_n$ an nBER.

<u>Claim 0:</u> Suppose that $T_0, T_1 \subseteq X$ are Borel transversals of F. Then $\mu(T_0) = \mu(T_1)$.

<u>pf(C0):</u> Enumerate $\Gamma = \{\gamma_k : k \in \omega\}$. Define Borel

$$A_k = \{x \in T_0 : k \text{ is least s.t. } \gamma_k \cdot x \in T_1 \cap [x]_F\}.$$

Then $T_0 = \bigsqcup_k A_k$ while $T_1 = \bigsqcup_k \gamma_k \cdot A_k$. ■(C0).

<u>Claim 1:</u> Any Borel transv. $T \subseteq X$ of F has $\mu(T) = \frac{1}{n}$.

<u>pf(C1):</u> Lusin-Novikov grants a partition of X into n-many Borel transversals of F. ■(C1)

<u>Claim 2:</u> For $A \subseteq X$ Borel, $\mu(A) = \int m_{[x]_F}(A)\, d\mu$.

<u>pf(C2):</u> For $i \leq n$ put $A_i = \{x \in X : |A \cap [x]_F| = i\}$, so for all $x \in A_i$ we have $m_{[x]_F}(A) = i/n$.

Then
$$\begin{aligned}\mu(A) &= \sum_{i \leq n} \mu(A \cap A_i) \\ &= \sum_{i \leq n} \frac{i}{n} \mu(A_i) \quad \text{[by Claim 1]} \\ &= \int m_{[x]_F}(A)\, d\mu. \end{aligned}$$
■(C2)

We did it! ■($\mathcal{L}$)

③ **Def:** Suppose that Γ acts on X.

ⓐ For finite $S \subseteq \Gamma$ and arbitrary $A \subseteq X$, the <u>S-interior</u> of A is $\mathrm{int}_S(A) = \{x \in A : S \cdot x \subseteq A\}$.

ⓑ For finite $S \subseteq \Gamma$ and $\varepsilon > 0$, an <u>(S, ε)-Følner set</u> is a finite non-$\emptyset$ $C \subseteq X$ satisfying

$$m_C(\mathrm{int}_S(C)) > 1 - \varepsilon.$$

ⓒ The action $\Gamma \curvearrowright X$ is <u>amenable</u> if (S, ε)-Følner sets exist for all choices of (S, ε) as above.

ⓓ The group Γ is <u>amenable</u> if all of its actions are amenable. This is equivalent to Γ having a free amenable action.

Examples:

- Finite groups are amenable: Γ itself is (S, ε)-Følner [for any (S, ε)] for the left-mult action $\Gamma \curvearrowright \Gamma$.
- $\mathbb{Z}$ is amenable: Given (S, ε), sufficiently long intervals will be (S, ε)-Følner.

Non-examples:

- The free group $F_2 = \langle a, b \rangle$ is not amenable. Consider $S = \{a^{\pm}, b^{\pm}\}$.

  <u>Claim:</u> There is no $(S, \frac{1}{2})$-Følner set for a free action $F_2 \curvearrowright X$.

  <u>pf(C):</u> Suppose we had such a set C. Then the graph G on C with $x \, G \, y$ iff $\exists \alpha \in S$ with $\alpha \cdot x = y$ would be a finite acyclic graph with more than half of its vertices having degree 4. No such G exists. ∎(C)

(4) **Thm:** Suppose that $\Gamma \curvearrowright X$ in a free μ-pres Borel fashion with orbit eq. rel E. If E is hyperfinite, then Γ is amenable.

pf: Let S be an arbitrary finite subset of Γ and $\varepsilon > 0$. We want to find an (S, ε)-Følner set in X.
Fix FBERs $(E_n)_{n \in \omega}$ witnessing hyperfiniteness of E.
Define for $n \in \omega$ a Borel set $A_n \subseteq X$ by

$$A_n = \{x \in X : S \cdot x \subseteq [x]_{E_n}\}.$$

Claim 0: $X = \bigcup_n A_n$.

pf(C0): For each $x \in X$, $S \cdot x$ is a finite subset of $[x]_E = \bigcup_n [x]_{E_n}$. Thus, $\exists n$ $S \cdot x \subseteq [x]_{E_n}$. ■(C0)

We may thus find $n \in \omega$ with $\mu(A_n) > 1 - \varepsilon$.
Applying our Lemma to $E_n \subseteq E$, we see

$$\mu(A_n) = \int m_{[x]_{E_n}}(A_n) \, d\mu(x) > 1 - \varepsilon.$$

In particular, we may fix $x \in X$ satisfying

$$m_{[x]_{E_n}}(A_n) > 1 - \varepsilon.$$

Put $C = [x]_{E_n}$.

Claim 1: C is (S, ε)-Følner.

pf(C1): By construction, $\mathrm{int}_S(C) = A_n \cap C$.

$$\begin{aligned} \text{So } m_C(\mathrm{int}_S(C)) &= m_C(A_n \cap C) \\ &= m_{[x]_{E_n}}(A_n) \\ &> 1 - \varepsilon \quad \text{as desired.} \end{aligned}$$ ■(C1)

We did it again! ■(Thm)

DST

Lecture 29

Generic hyperfiniteness

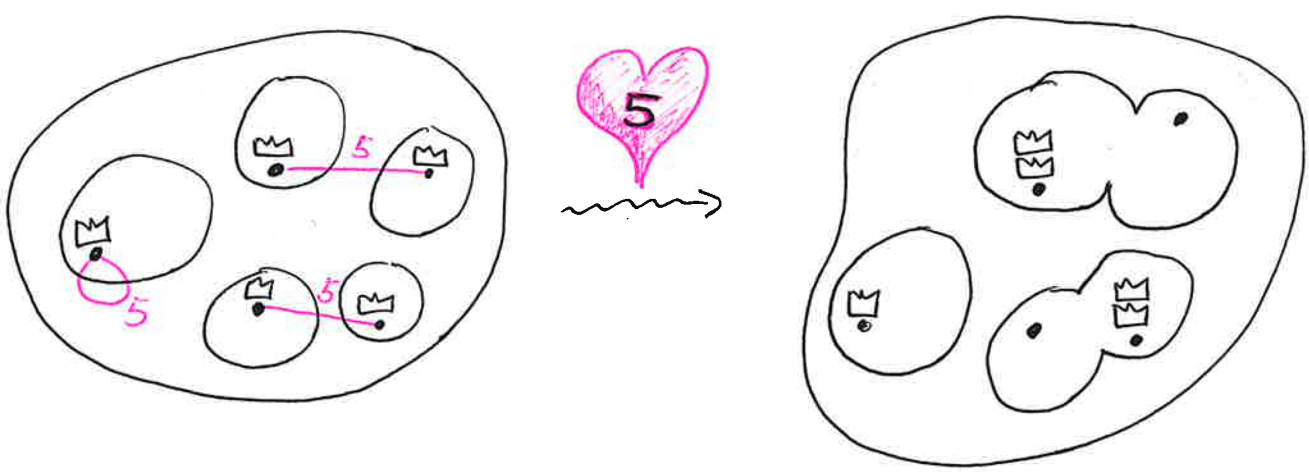
Last time, we experienced an intrusion of measure theory:

Thm: Suppose that Γ is a countable non-amenable group acting in a (μ-a.e.) free, μ-preserving Borel fashion on a standard probability space (X, μ). Then its orbit equivalence relation is NOT hyperfinite.

Today, we argue that our tried-and-true method of Baire category cannot detect non-hyperfiniteness:

Thm (Hjorth-Kechris, 1996, building on Sullivan-Weiss-Wright, Woodin) Suppose that E is a CBER on a Polish space X. Then there is an E-invariant comeager Borel set $C \subseteq X$ such that $E \upharpoonright C$ is hyperfinite.

Idea: "Parametrize attempts" at witnessing hfness, and analyze where these attempts succeed.



② pf (Thm):

For convenience, fix a Borel linear order $<$ on X.
By Feldman-Moore, fix Borel involutions $\iota_n : X \to X$ with $\bigcup_n \iota_n = E$. We recursively define for each $s \in \omega^{<\omega}$ an FBER F_s on X:

- $F_\emptyset = \Delta_X$
- $x \, F_{s^\frown n} \, y$ iff $x \, F_s \, y$ OR $\iota_n(\tilde{x}) = \tilde{y}$,
  where $\tilde{x}$ = $<$-largest elt of $[x]_{F_s}$
  $\tilde{y}$ = $<$-largest elt of $[y]_{F_s}$.

Observe that $F_s \subseteq F_{s^\frown n} \subseteq E$.

Now, for each $\sigma \in \omega^\omega$ we put $E_\sigma = \bigcup_n F_{\sigma \upharpoonright n}$.
Each E_σ is hyperfinite by fiat, and $E_\sigma \subseteq E$.
We also put $C_\sigma = \{ x \in X : [x]_{E_\sigma} = [x]_E \}$.

Claim 0: The assignments $\sigma \mapsto E_\sigma$, $\sigma \mapsto C_\sigma$ are uniformly Borel in the sense that

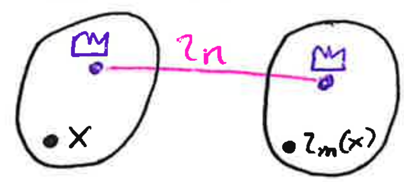
$$P = \{ (x,y,\sigma) \in X \times X \times \omega^\omega : x \, E_\sigma \, y \} \text{ and}$$
$$Q = \{ (x,\sigma) \in X \times \omega^\omega : x \in C_\sigma \} \text{ are Borel.}$$

pf (C0): Write $P = \bigcup \{ F_s \times N_s : s \in \omega^{<\omega} \}$, after harmlessly identifying $((x,y),\sigma)$ with (x,y,σ).
Then $(x,\sigma) \in Q$ iff $\forall m \; (x, \iota_m(x), \sigma) \in P$. ∎(C0)

③ pf (Thm, cont.)

Claim 1: For each $x \in X$, the vertical section
$$Q_x = \{\sigma \in \omega^\omega : x \in C_\sigma\}$$
is comeager in ω^ω.

pf (C1): Putting $A_{x,m} = \{\sigma \in \omega^\omega : x \; E_\sigma \; \gamma_m(x)\}$, we see $Q_x = \bigcap_m A_{x,m}$. It therefore suffices to show that each $A_{x,m}$ contains an open dense set. For this, it suffices to show for all $s \in \omega^{<\omega}$ that $N_s \cap A_{x,m}$ has non-$\emptyset$ interior. We now examine $[x]_{F_s}$ and $[\gamma_m(x)]_{F_s}$:



As illustrated, we may find $n \in \omega$ so that $x \; F_{s^\frown n} \; \gamma_m(x)$. Then $N_{s^\frown n} \subseteq N_s \cap A_{x,m}$. ■ (C1)

We may rewrite the prior claim as
$$\forall x \in X \; \forall^* \sigma \in \omega^\omega \;\; x \in C_\sigma.$$
As Q is BP, we may invoke Kuratowski-Ulam:
$$\forall^* \sigma \in \omega^\omega \; \forall^* x \in X \;\; x \in C_\sigma.$$
Fixing any σ in this comeager set, we see that C_σ is an E-invariant comeager Borel set. Moreover, $E \restriction C_\sigma = E_\sigma \restriction C_\sigma$, so in particular we see that $E \restriction C_\sigma$ is hyperfinite as desired. ■ (Thm)

DST

Lecture 30

The Galvin-Prikry theorem (part 1)

Def: Given a set B, we let
$$[B]^\omega = [B]^{\aleph_0} = \{A \subseteq B : |A| = \aleph_0\}.$$

Def: Ramsey space is $[\omega]^\omega$. We may identify it with $\{x \in 2^\omega : \exists^\infty n \;\, x(n) = 1\}$. Thus, $[\omega]^\omega$ has a std Borel structure inherited from 2^ω.

Thm (Galvin-Prikry, 1973):
Suppose that $k \in \omega$ and $c : [\omega]^\omega \to k$ is Borel. Then there is $B \in [\omega]^\omega$ s.t. $c \restriction [B]^\omega$ is constant.

Following Ellentuck (1974), we shall prove this using a non-Polish topology generating a bigger Borel σ-algebra.

Def: The Ellentuck topology on $[\omega]^\omega$ is generated by
- $\mathcal{N}_s \cap [\omega]^\omega$ for $s \in 2^{<\omega}$
- $[B]^\omega$ for $B \in [\omega]^\omega$

So a typical basic open set is
$$[s, B] = \mathcal{N}_s \cap [B]^\omega = \{A \in [\omega]^\omega : s \sqsubseteq A \text{ and } A \subseteq B\}.$$

$$\underbrace{\underline{0}\ \underline{1}\ \underline{1}\ \underline{0}\ \underline{0}}_{s}\ \underline{?}\ \underline{0}\ \underline{0}\ \underline{?}\ \underline{0}\ \underline{?}\ \underline{0}\ \ldots$$

B

Remark: This topology is terribly non-separable. If $\{B_x : x \in 2^\omega\}$ is an almost disjoint family of elements of $[\omega]^\omega$, then $\{[B_x]^\omega : x \in 2^\omega\}$ is a pairwise disjoint family of non-$\emptyset$ open sets.

Today, we focus on a combinatorial lemma that will do some heavy lifting for us.

Def: Fix $\mathbb{X} \subseteq [\omega]^\omega$, and consider $s \in 2^{<\omega}$, $B \in [\omega]^\omega$.

ⓐ B accepts s (into $\mathbb{X}$) if $[s, B]$ is non-$\emptyset$ and $[s, B] \subseteq \mathbb{X}$.

ⓑ B rejects s (from $\mathbb{X}$) if no $A \in [B]^\omega$ accepts s.

Remarks: Let's connect these with the Ellentuck top.

ⓐ B accepts s into $\mathbb{X}$ exactly when $[s, B]$ witnesses that $[B]^\omega \cap \mathbb{X}$ has non-$\emptyset$ interior

ⓑ Suppose that B rejects all of its finite subsets, i.e., for all $s \in 2^{<\omega}$ and $A \in [s, B]$ we have $[s, A] \nsubseteq \mathbb{X}$. Then $[B]^\omega \cap \mathbb{X}$ has empty interior.

Key Lemma: Suppose that $\mathbb{X} \subseteq [\omega]^\omega$. One holds:

$\boxed{\text{I}}$ There is $A \in [\omega]^\omega$ that accepts $\emptyset$

$\boxed{\text{II}}$ There is $A \in [\omega]^\omega$ that rejects all of its finite subsets.

(3) pf (Key Lemma)

Indexing should be fixed...

Suppose that $\boxed{\text{I}}$ fails, i.e., that ω rejects $\emptyset$. We shall recursively construct $B_n \in [\omega]^\omega$, $a_n \in \omega$ so that

- $B_n \supseteq B_{n+1}$
- $a_n < a_{n+1}$
- $\{a_1, \dots, a_n\} \subseteq B_n$
- B_n rejects every subset of $\{a_1, \dots, a_{n-1}\}$.

Would be better to directly use elements of $2^{<\omega}$ here.

Suspend disbelief and assume we can pull this off. Put $A = \{a_{n+1} : n \in \omega\}$, noting for all n that $A \subseteq B_n$.

Claim 0: A rejects all of its finite subsets.

pf (C0): Consider $s \subseteq \{a_1, \dots, a_{n-1}\}$. By construction, B_n rejects s, and thus A rejects s as well. $\blacksquare$(C0).

We now perform the recursive construction.

Stage 0: Put $B_0 = \omega$, which rejects every subset of $\emptyset$.

Stage $n+1$: Suppose we have B_n and $\bar{a}_n = \{a_i : i < n\}$ as above. We want to find:

- $a_n \in B_n$ with $a_n > a_{n-1}$
- $B_{n+1} \subseteq B_n$ with $\bar{a}_n \cup \{a_n\} \subseteq B_{n+1}$

such that B_{n+1} rejects all subsets of $\bar{a}_n \cup \{a_n\}$. Towards a contradiction, suppose no such a_n and B_{n+1} exist. We shall recursively construct $C_k \in [B_n]^\omega$, $b_k \in \omega$, and $s_k \subseteq \bar{a}_n$ such that

- $C_k \supseteq C_{k+1}$
- $b_k < b_{k+1}$
- $\bar{a}_n \cup \{b_0, \dots, b_{k+1}\} \subseteq C_k$.
- C_k accepts $s_k \cup \{b_k\}$.

④ pf (Key Lemma, cont.)

Stage 0: Choose any $b_0 > a_{n-1}$ in B_n. By assumption, B_n does not reject all subsets of $\bar{a}_n \cup \{b_0\}$, but it rejects all subsets of $\bar{a}_n$. Thus, there is some $C \in [B_n]^\omega$ and $s_0 \subseteq \bar{a}_n$ so that C_0 accepts $s_0 \cup \{b_0\}$.

Stage k+1: Choose any $b_{k+1} > b_k$ in C_k. By assumption, C_k does not reject all subsets of $\bar{a}_n \cup \{b_{k+1}\}$, but it rejects all subsets of $\bar{a}_n$. As before, there is some $C_{k+1} \in [C_k]^\omega$ and $s_{k+1} \subseteq \bar{a}_n$ so that C_{k+1} accepts $s_{k+1} \cup \{b_{k+1}\}$.

This completes the recursive (sub)construction. Now, there are only finitely many possible values of s_k, so fix some infinite $K \subseteq \omega$ so that $\forall k \in K \;\; s_k = s$. Put $B = \bar{a}_n \cup \{b_k : k \in K\}$.

Claim 1: B accepts s.

pf (C1): Consider an arbitrary $C \in [s, B]$.

Let $k \in K$ be least with $b_k \in C$. Then

$C \in [s \cup \{b_k\}, B] \subseteq [s \cup \{b_k\}, C_k] \subseteq \mathcal{X}$. ■ (C1)

This claim contradicts the fact that B_n rejects every subset of $\bar{a}_n$ (in particular, s). Having hit a contradiction, we conclude that the original recursive construction can be completed. Claim 0 then grants $\boxed{\text{II}}$. ■ (K.L.)

DST

Lecture 31

The Galvin-Prikry theorem (part 2)

Recall: The Ellentuck topology on $[\omega]^\omega$ with basic open sets $[s,B] = \{A \in [\omega]^\omega : s \sqsubseteq A \text{ and } A \subseteq B\}$ for $s \in 2^{<\omega}$ and $B \in [\omega]^\omega$.

Key Lemma: Suppose that $\mathbb{X} \subseteq [\omega]^\omega$. One holds:

$\boxed{\text{I}}$ There is $A \in [\omega]^\omega$ with $[A]^\omega \subseteq \mathbb{X}$

$\boxed{\text{II}}$ There is $A \in [\omega]$ s.t. $[A]^\omega \cap \mathbb{X}$ has empty interior.

Defs: Suppose that $\mathbb{X} \subseteq [\omega]^\omega$.

(a) $\mathbb{X}$ is Ramsey if there is $A \in [\omega]^\omega$ s.t. either

- $[A]^\omega \subseteq \mathbb{X}$, or
- $[A]^\omega \cap \mathbb{X} = \emptyset$

Galvin-Prikry is equivalent to "Borel sets are Ramsey."

(b) $\mathbb{X}$ is completely Ramsey if for all non-empty basic open $[s,B] \subseteq [\omega]^\omega$, there is $A \in [s,B]$ s.t. either

- $[s,A] \subseteq \mathbb{X}$, or
- $[s,A] \cap \mathbb{X} = \emptyset$.

(c) $\mathbb{X}$ is Ramsey null if for all non-empty $[s,B]$ the second alternative holds. I.e., there is $A \in [s,B]$ with $[s,A] \cap \mathbb{X} = \emptyset$.

② **Prop:** The collection of Ramsey null subsets of $[\omega]^\omega$ forms a σ-ideal.

pf: Suppose that $(X_n)_{n\in\omega}$ is a sequence of R-null subsets of $[\omega]^\omega$. We want $\bigcup_n X_n$ R-null.

Fix non-$\emptyset$ $[s, B]$ with the hope of finding $A \in [s, B]$ with $[s, A] \cap \bigcup_n X_n = \emptyset$.

We shall recursively construct $s_n \in 2^{<\omega}$ and $B_n \in [\omega]^\omega$ so that
- $s_n \subsetneq s_{n+1}$ (s_{n+1} has more 1s)
- $B_{n+1} \in [s_n, B_n]$
- for all subsets $t \subseteq s_n$, $[t, B_{n+1}] \cap X_n = \emptyset$.

Stage 0: Put $s_0 = s$ and $B_0 = B$. ☺

Stage n+1: Given s_n and B_n, let $(t_k)_{k<m}$ enumerate the subsets of s_n, and recursively build a decreasing sequence $C_k \in [s_n, B_n]$ with $[t_k, C_k] \cap X_n = \emptyset$ (using that X_n is R-null).

Put $B_{n+1} = C_{k-1}$, and $s_{n+1} = s_n 00..01$
$= s_n \cup \{b\}$ some $b \in B_{n+1}$.

This completes the recursive construction. Put

Claim: $[s, A] \cap \bigcup_n X_n = \emptyset$

pf(c): For fixed $n \in \omega$, we know that
$$[s, A] \subseteq \bigcup_{t \subseteq s_n} [t, B_{n+1}]$$
and thus $[s, A] \cap X_n = \emptyset$. ■ (Prop)

③ **Thm** (Ellentuck): If $\mathbb{X} \subseteq [\omega]^\omega$ is Ellentuck-BP, then $\mathbb{X}$ is completely Ramsey.

pf: We proceed in several steps.

Step A: If $\mathbb{X}$ is open, it is Ramsey.

pf(A): Apply our Key Lemma to $\mathbb{X}$:

Case I: $[A]^\omega \subseteq \mathbb{X}$. ☺ ✓

Case II: $[A]^\omega \cap \mathbb{X}$ has empty interior. It is open, thus it is empty. ✓ ∎(A)

Step B: If $\mathbb{X}$ is open, it is completely Ramsey.

pf(B): Fix non-empty $[s, B] \subseteq [\omega]^\omega$, and let $\varphi : [\omega]^\omega \longrightarrow [s, B]$ be the homeomorphism

$$a_0\ a_1\ a_2 \dots \longmapsto \underbrace{0\ 1\ 1}_{s}\ 0\ \underbrace{a_0\ 0\ a_1\ a_2}_{B}\ 0 \ .$$

Now $[s, B] \cap \mathbb{X}$ is open, so its φ-preimage is Ramsey. Witness this by $A \in [\omega]^\omega$, and check that $\varphi(A)$ works for $[s, B]$. ∎(B)

Step C: If $\mathbb{X}$ is closed, it is completely Ramsey.

pf(C): ☺ ∎(C)

Step D: If $\mathbb{X}$ is nowhere dense, it is Ramsey null.

pf(D): We may assume $\mathbb{X}$ is closed with empty interior. Fix non-empty $[s, B]$. $\mathbb{X}$ is completely Ramsey, so there is $A \in [s, B]$ s.t.

- $[s, A] \subseteq \mathbb{X}$, or
- $[s, A] \cap \mathbb{X} = \emptyset$

The first option cannot occur. ∎(D)

④ pf (Thm, cont.)

Step E: If $\mathcal{X}$ is meager, it is Ramsey null.

pf (E): This follows from step D and our earlier proposition. ■(E)

Step F: If $\mathcal{X}$ is BP, it is completely Ramsey.

pf(F): Write $\mathcal{X} = \mathcal{Y} \triangle W$ with $\mathcal{Y}$ open and W meager. Fix non-empty $[s, B]$. First, use step E to find $B' \in [s, B]$ so that $[s, B'] \cap W = \emptyset$. Next, use step B to find $A \in [s, B']$ such that either

- $[s, A] \subseteq \mathcal{Y}$, or
- $[s, A] \cap \mathcal{Y} = \emptyset$.

As $[s, A] \cap W = \emptyset$, we conclude that either

- $[s, A] \subseteq \mathcal{X}$ or
- $[s, A] \cap \mathcal{X} = \emptyset$

as desired. ■(F) ■(Thm)

Remarks: Various assertions here have converses. Suppose $\mathcal{X} \subseteq [\omega]^\omega$. Then

- $\mathcal{X}$ is BP iff $\mathcal{X}$ is completely Ramsey
- $\mathcal{X}$ is nwdense iff $\mathcal{X}$ is meager iff $\mathcal{X}$ is R-null.

This boils down to showing R-null $\Rightarrow$ nwdense, which follows from our Key Lemma.

①

DST

Lecture 32

Rosenthal's ℓ^1 dichotomy (part 1)

Today's setup: S is some set, $\forall n \in \omega$ we have sets $A_n, B_n \subseteq S$ with $A_n \cap B_n = \emptyset$.

Def: We say that the sequence $(A_n, B_n)_{n\in\omega}$ is independent if for all finite disjoint $F, G \subseteq \omega$

$$\bigcap_{n\in F} A_n \cap \bigcap_{n \in G} B_n \neq \emptyset.$$

Def: The sequence $(A_n, B_n)_{n\in\omega}$ is convergent if $\forall s \in S$ one of $\{n\in\omega : s \notin A_n\}$ / $\{n\in\omega : s\notin B_n\}$ is cofinite

Negation: $\exists s \in S$ with both $\{n\in\omega : s\in A_n\}$ / $\{n\in\omega : s \in B_n\}$ infinite.

Motivation: Suppose we have $f_n : S \to \mathbb{R}$ and we want to test whether f_n converges (ptwise). If we fix disjoint closed $C, D \subseteq \mathbb{R}$ and put

$$A_n = \{s\in S : f_n(s) \in C\}$$
$$B_n = \{s \in S : f_n(s) \in D\}$$

we can ask about convergence of $(A_n, B_n)_{n\in\omega}$. If not convergent, then some $s\in S$ is in infinitely many A_n and B_n, precluding convergence of $f_n(s)$.

② **Example:** $S = \{x \in 2^\omega : x \text{ is eventually } 0\}$

$$A_n = \{x \in S : x(n) = 0\}$$

$$B_n = \{x \in S : x(n) = 1\}$$

This (A_n, B_n) is both independent and convergent.

Prop: Suppose that $(A_n, B_n)_{n\in\omega}$ is a sequence of disjoint pairs of subsets of S. Then some subsequence is independent or convergent.

pf: We define $\mathbb{X} \subseteq [\omega]^\omega$ as follows: represent elements of $[\omega]^\omega$ by increasing enumerations, and put $\{n_i : i \in \omega\} \in \mathbb{X}$ iff $\forall k \in \omega$

$$\bigcap\{A_{n_i} : i<k \text{ even}\} \cap \bigcap\{B_{n_i} : i<k \text{ odd}\} \neq \emptyset$$

Ex: Consider $\{\overset{n_0}{5}, \overset{n_1}{7}, \overset{n_2}{8}, \overset{n_3}{9}, \overset{n_4}{11}, \overset{n_5}{17}, \ldots\} \in [\omega]^\omega$

For $k=5$ we ask $(A_5 \cap A_8 \cap A_{11}) \cap (B_7 \cap B_9) \overset{?}{=} \emptyset$.

The set $\mathbb{X}$ is (Cantor-) closed, and so the Galvin-Prikry theorem applies. Fix $Z \in [\omega]^\omega$ s.t. either $\boxed{\mathrm{I}}$ $[Z]^\omega \subseteq \mathbb{X}$, or

$\boxed{\mathrm{II}}$ $[Z]^\omega \cap \mathbb{X} = \emptyset$.

Write $Z = \{m_i : i \in \omega\}$ in increasing enumeration.

③ pf (Prop, cont.)

Claim I: If $[Z]^\omega \subseteq \mathbb{X}$, then the subsequence

$(A_{m_{2i+1}}, B_{m_{2i+1}})$ is independent.

pf (CI): Fix disjoint finite $F, G \subseteq ODD$.

We want $\bigcap_{i\in F} A_{m_i} \cap \bigcap B_{m_i} \neq \emptyset$.

Build a subset of Z so that F appears at even indices and G at odd indices:

$$\begin{array}{lcccccccc} \text{even:} & F & & F & & ? & & F & \\ \text{odd:} & & ? & & G & & G & & \text{etc.} \end{array}$$

For example, say $F = \{1, 3, 9\}$

$G = \{5, 7, 11\}$,

take $\{\underset{F}{m_1}, m_2, \underset{F}{m_3}, \underset{G}{m_5}, m_6, \underset{G}{m_7}, \underset{F}{m_9}, \underset{G}{m_{11}}, \ldots\} \subseteq Z$

The result is in $\mathbb{X}$ by hypothesis, granting independence of $(A_{m_{odd}}, B_{m_{odd}})$. ■(CI)

Claim II: If $[Z]^\omega \cap \mathbb{X} = \emptyset$, then the sequence

(A_{m_i}, B_{m_i}) is convergent.

pf (CII): Suppose not, and fix $s \in S$ and infinite disjoint $I = \{i \in \omega : s \in A_{m_i}\}$, $J = \{i \in \omega : s \in B_{m_i}\}$. Build $\mathbb{Y} \subseteq Z$ whose increasing enumeration alternates between I, J. Then $\mathbb{Y} \in \mathbb{X}$ as s inhabits all required intersections, contradicting the hypothesis. ■(CII)

■(Prop)

④ It will be convenient to upgrade this slightly.

Prop: Suppose for each $m \in \omega$ we have a sequence $(A_n^m, B_n^m)_{n\in\omega}$ of disjoint pairs of subsets of S. At least one holds:

$\boxed{\text{I}}$ There is a subsequence $(n_i)_{i\in\omega}$ s.t. $\exists m \in \omega$ $(A_{n_i}^m, B_{n_i}^m)$ is independent.

$\boxed{\text{II}}$ There is a subsequence $(n_i)_{i\in\omega}$ s.t. $\forall m \in \omega$ $(A_{n_i}^m, B_{n_i}^m)$ is convergent.

pf: Suppose $\boxed{\text{I}}$ fails; we achieve $\boxed{\text{II}}$ by diagonalization. First find a subsequence n_i^0 with $(A_{n_i^0}^0, B_{n_i^0}^0)$ convergent, using our previous Proposition. Find a further subsequence n_i^1 with $(A_{n_i^1}^1, B_{n_i^1}^1)$ convergent. Continue in this fashion, and then check that the diagonal sequence $(n_i^i)_{i\in\omega}$ is as desired in $\boxed{\text{II}}$.

■ (Prop)

DST

Lecture 33

Rosenthal's ℓ^1 dichotomy (part 2)

Def: A (real) Banach space is an $\mathbb{R}$-vector space equipped with a norm so that the induced metric $d : (x,y) \mapsto \|x-y\|$ is complete.

Examples: Fix a set S.

① $\ell^1(S) = \{f \in \mathbb{R}^S : \sum_s |f(s)| < \infty\}$ with coord-wise operations. The norm is $\|f\|_1 = \sum_s |f(s)|$. We write ℓ^1 for $\ell^1(\omega) \subseteq \mathbb{R}^\omega$.

⓪⓪ $\ell^\infty(S) = \{f \in \mathbb{R}^S : \sup_s |f(s)| < \infty\}$ with coord-wise operations. The norm is $\|f\|_\infty = \sup_s |f(s)|$.

Def: A sequence $(x_n)_{n\in\omega}$ in a Banach space X is weakly Cauchy if for every continuous linear map $T : X \to \mathbb{R}$ the sequence $(T(x_n))_{n\in\omega}$ converges in $\mathbb{R}$.

Remarks:

ⓐ If X and its dual are separable, then every (norm-) bounded sequence in X admits a weakly Cauchy subsequence.

ⓑ In ℓ^1, the "standard basis" sequence $(\chi_{\{n\}})_{n\in\omega}$ admits no weakly Cauchy subsequence.

Remarkably, ℓ^1 is the canonical impediment to this!

② Thm (Rosenthal, 1974):

Suppose that X is a (real) Banach space. Exactly one:

$\boxed{\mathrm{I}}$ Every bounded sequence in X admits a weakly Cauchy subsequence.

$\boxed{\mathrm{II}}$ There is a subspace of X isomorphic to ℓ^1.

Caveat: "Isomorphic" in the sense of topological vector spaces. Not necessarily "isometrically isomorphic."

Let's convert this into a more combinatorial form

Def: Given a real Banach space X, we say that a sequence $(x_n)_{n\in\omega}$ is ℓ^1-ish if

$$\exists a, b > 0 \quad \forall N \in \omega \quad \forall c_n \in \mathbb{R}$$
$$a \cdot \sum_{n<N} |c_n| \leq \Big\| \sum_{n<N} c_n x_n \Big\|_X \leq b \cdot \sum_{n<N} |c_n|.$$

Remark: If $(x_n)_{n\in\omega}$ is ℓ^1-ish, then the map $f \mapsto \sum_n f(n) x_n$ yields an isomorphism from ℓ^1 to its image.

So it suffices to prove

Thm* (Rosenthal):

Suppose that $(f_n)_{n\in\omega}$ is a bounded sequence in $\ell^\infty(S)$. Then it admits a subsequence (f_{n_k}) s.t. one of

$\boxed{\mathrm{I}}$ (f_{n_k}) is pointwise convergent

$\boxed{\mathrm{II}}$ (f_{n_k}) is ℓ^1-ish.

Why? Take $S = \{ T : X \to \mathbb{R} \text{ linear} : \forall x \ \ |T(x)| \leq \|x\| \}$

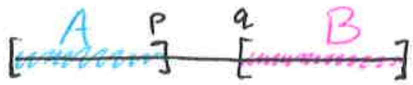
$f_n : T \mapsto T(x_n)$

③ pf(Thm*):

Given our bounded sequence (f_n) in $\ell^\infty(S)$, we define for rational $p<q$ sets $A_n^{pq}, B_n^{pq} \subseteq S$:

$$A_n^{pq} = \{s \in S : f_n(s) \le p\},$$
$$B_n^{pq} = \{s \in S : f_n(s) \ge q\}$$

Certainly, $A_n^{pq} \cap B_n^{pq} = \emptyset$.



By Lecture 32, we may pass to a subsequence satisfying one of

$\boxed{\text{I}}$ $\forall p,q$ (A_n^{pq}, B_n^{pq}) is convergent, or

$\boxed{\text{II}}$ $\exists p,q$ (A_n^{pq}, B_n^{pq}) is independent.

Claim I: $\forall p,q$ (A_n^{pq}, B_n^{pq}) conv $\Rightarrow$ (f_n) ptwise conv.

pf(CI): Exercise. ☺(CI)

Claim II: $\exists p,q$ (A_n^{pq}, B_n^{pq}) indep $\Rightarrow$ (f_n) ℓ^1-ish.

pf(CII): Fix $b<\infty$ s.t. all $\|f_n\|_\infty < b$.

Then $\left\| \sum_{n<N} c_n f_n \right\|_\infty \le b \sum_{n<N} |c_n|$. Yay! Halfway!

We shall declare $a = \frac{q-p}{2}$ and hope that $a \cdot \sum_{n<N} |c_n| \le \left\| \sum_{n<N} c_n f_n \right\|_\infty$.

④ pf (Thm*, cont.)

pf (CII, cont.)

Put $F = \{n<N : c_n \geq 0\}$

$G = \{n<N : c_n < 0\}$

By independence of (A_n, B_n), we find

$$s \in \bigcap_{n\in F} A_n \cap \bigcap_{n\in G} B_n$$

$$t \in \bigcap_{n\in G} A_n \cap \bigcap_{n\in F} B_n .$$

We then compute

$$\Sigma_n c_n f_n(s) \leq \sum_{n\in F} |c_n| p - \sum_{n\in G} |c_n| q$$

$$\Sigma_n c_n f_n(t) \geq \sum_{n\in F} |c_n| q - \sum_{n\in G} |c_n| p .$$

So

$$2 \|\Sigma_n c_n f_n\|_\infty \geq \Sigma_n c_n f_n(t) - \Sigma_n c_n f_n(s)$$

$$\geq \Sigma_n q|c_n| - \Sigma_n p|c_n|$$

$$= (q-p) \Sigma_n |c_n| .$$

In other words,

$$\frac{q-p}{2} \Sigma_n |c_n| \leq \|\Sigma_n c_n f_n\|_\infty$$

establishing our desired ℓ^1-ishness. ■ (CII)

■ (Thm*)

DST

Lecture 34

Graphs associated with functions (part 1)

Defs: Suppose that $f: X \to X$ is a partial function

ⓐ G_f is the smallest graph containing $f \setminus \Delta_X$.

ⓑ $E_f = E_{G_f}$ is the smallest eq. rel. containing f.

Remark: The sets $\{x \in X : f(x) = x\}$ and $X \setminus \mathrm{dom}(f)$ meet each E_f-class in at most one point. It is thus typically harmless to assume that f is total and fixed-point free. With these assumptions, we have:

- $x \, G_f \, y$ iff $f(x) = y$ or $f(y) = x$
- $x \, E_f \, y$ iff $\exists m, n \in \omega \; f^m(x) = f^n(y)$.

Remark: When X is std Borel and $f: X \to X$ is Borel, then so are G_f and E_f.

Examples:

① If $f: X \to X$ is a Borel injection, then G_f has degree bounded by 2. So $\chi_B(G_f) \leq 3$.

$\sqrt{2}$ From $\boxed{\text{HW}}$, the irrational rotation yields an example of injective Borel f satisfying $\chi_B(G_f) = 3$.

② Examples (cont.)

② We examine the shift on Ramsey space $[\omega]^\omega$, viewed as a std Borel subspace of 2^ω.

$$s : A \mapsto A \setminus \{\min A\}$$

Prop: $\chi_B(G_s) = \aleph_0$.

pf: First observe that $c : A \mapsto \min A$ is a Borel proper $\aleph_0$-coloring of G_s. It thus suffices to argue that for $k \in \omega$, no Borel function $c : [\omega]^\omega \to k$ is a proper coloring of G_s. Galvin-Prikry yields $A \in [\omega]^\omega$ so that $c \restriction [A]^\omega$ is constant. In particular, $c(A) = c(s(A))$, precluding c from being a proper coloring. ∎(Prop)

Remark: The above argument applies to the wider class of Ellentuck-BP colorings.

Today we shall discuss a few results from Kechris - Solecki - Todorcevic (1999) illustrating that the above examples are fully representative of graphs induced by Borel functions.

Thm A (KST): Suppose that $f : X \to X$ is a fixed-pt free Borel function on std Borel X. Then $\chi_B(G_f) \leq \aleph_0$.

③ pf (Thm A)

Observe first that for any $B \subseteq X$, the set $B \setminus f^{-1}(B)$ is G_f-independent, as

$$x \in B \setminus f^{-1}(B) \Rightarrow f(x) \notin B.$$

Fix now a countable generating algebra $\mathcal{A}$ for the Borel σ-algebra on X. We consider the countable collection $\{A \setminus f^{-1}(A) : A \in \mathcal{A}\}$ of Borel G_f-independent sets.

Claim: The family $\{A \setminus f^{-1}(A) : A \in \mathcal{A}\}$ covers X.

pf(C): For each $x \in X$, since $f(x) \neq x$ we may find $A \in \mathcal{A}$ s.t. $x \in A$ and $f(x) \notin A$. ■(C)

This cover witnesses $\chi_B(G_f) \leq \aleph_0$. ■(Thm A)

Def: Given $f : X \to X$ and $B \subseteq X$, we say that B is forward recurrent if $\forall x \in X \ \exists n \in \omega \ f^n(x) \in B$. Equivalently, $\bigcup_n f^{-n}(B) = X$.

Thm B (KST): Suppose that $f : X \to X$ is a fixed-pt free Borel function on std Borel X.

TFAE: $\boxed{\text{I}}$ $\chi_B(G_f) \leq 3$

$\boxed{\text{II}}$ $\chi_B(G_f) < \aleph_0$

$\boxed{\text{III}}$ There is a Borel fwd recurrent G_f-independent set.

Remark: We may weaken $\boxed{\text{III}}$ to:

$\boxed{\text{III}'}$ $\exists$ Borel $B \subseteq X$ with B, $X \setminus B$ both fwd recurrent, as then $B \setminus f^{-1}(B)$ satisfies $\boxed{\text{III}}$.

④ pf (Thm B):

$\boxed{\mathrm{I}} \Rightarrow \boxed{\mathrm{II}}$: ☺ ✓

$\boxed{\mathrm{II}} \Rightarrow \boxed{\mathrm{III}}$: Fix $k \in \omega$ and a Borel coloring $c : X \to k$ of G_f. Define Borel $b : X \to k$ by

$b : x \mapsto$ least $i < k$ s.t. $\exists^\infty n \in \omega$ $c(f^n(x)) = i$.

Observe for all $x \in X$ that $b(f(x)) = b(x)$.

Define Borel $B \subseteq X$ by $B = \{x \in X : c(x) = b(x)\}$.

This B is fwd recurrent by definition of b.

B is G_f-indep as $b(f(x)) = b(x)$ while $c(f(x)) \neq c(x)$, so at most one of $x, f(x)$ is in B. ✓

$\boxed{\mathrm{III}} \Rightarrow \boxed{\mathrm{I}}$: Fix Borel $B \subseteq X$ that is fwd recurrent and G_f-independent. Define Borel $d : X \to \omega$ by $d : x \mapsto$ least $n \in \omega$ with $f^n(x) \in B$.

Next, define Borel $c : X \to 3$ by

$$x \longmapsto \begin{cases} d(x) \bmod 2, & \text{if } x \notin B \\ 2 & \text{if } x \in B. \end{cases}$$

<u>Claim:</u> This c is a proper coloring of G_f.

<u>pf (C):</u> Given $x \in X$, at most one of $x, f(x)$ is in B. If neither is in B, then

$$d(x) = d(f(x)) + 1.$$

Either way, $c(f(x)) \neq c(x)$ as desired. ■(C) ✓

■(Thm B)

DST

Lecture 35

Graphs associated with functions (part 2)

Defs: Suppose that X is std Borel and $f: X \to X$ is Borel.

ⓐ The forward orbit of $x \in X$ is the set
$$f^\omega(x) = \{f^n(x) : n \in \omega\}.$$

ⓑ We say that f is aperiodic if every forward orbit is infinite. Equiv, $\forall x \in X\ \forall n > 0\ f^n(x) \neq x$.

Remarks:

ⓐ A set is forward recurrent (for f) exactly when it meets every forward orbit.

ⓑ For any Borel $f: X \to X$, the set of periodic points $\{x \in X : \exists n > 0\ f^n(x) = x\}$ meets each E_f-class in a finite set. We may thus regard any such f as aperiodic "off a smooth set."

Marker Lemma: Suppose that X is std Borel and that $f: X \to X$ is an aperiodic Borel function. Then f admits a vanishing sequence of fwd recurrent Borel markers. More precisely, there are Borel sets $A_n \subseteq X$ satisfying:

- $A_{n+1} \subseteq A_n$
- $\bigcap_n A_n = \emptyset$
- A_n is forward recurrent for f.

pf (ML, sketch)

The argument is very similar to the Marker Lemma for aperiodic CBERs from Lecture 27.
WLOG $X = 2^\omega$. We define for each $n \in \omega$ a Borel E_f-invariant function $s_n : 2^\omega \to 2^n$ via

$$x \mapsto \text{least } t \in 2^n \text{ with } f^\omega(x) \cap N_t \text{ infinite.}$$

Define Borel $A_n' = \{x \in 2^\omega : x \in N_{s_n(x)}\}$. Check that $A_\omega' = \bigcap_n A_n'$ meets each E_f-class in at most one point, and declare $A_n = A_n' \setminus A_\omega'$.

■ (ML, sketch)

Def: A (Borel) equivalence relation is hypersmooth if it is an increasing union of smooth eq. rels.

Cor: Given any Borel function $f : X \to X$ on std Borel X, the induced equivalence relation E_f is hypersmooth.

pf: WLOG, we may assume that f is aperiodic. Fix a vanishing sequence $(A_n)_{n\in\omega}$ of fwd recurrent Borel markers. Define for each $n \in \omega$ a Borel function $g_n : X \to A_n$ via

$$g_n : x \mapsto f^k(x), \ k \text{ least s.t. } f^k(x) \in A_n.$$

Finally, define E_n by

$$x \, E_n \, y \text{ iff } g_n(x) = g_n(y).$$

These are smooth by construction, and witness hypersmoothness of E. ■ (Cor)

③ **Thm** (Miller, 2008): Suppose that $f: X \to X$ is a Borel function on Polish X. Then $\chi_{BP}(G_f) \leq 3$. More precisely, there is an E_f-invariant comeager Borel $Y \subseteq X$ so that $\chi_B(G_f \upharpoonright Y) \leq 3$.

Remark: A nearly identical argument yields $\chi_\mu(G_f) \leq 3$ for any Borel probability measure μ on X.

pf (Thm): The periodic part of f is easy to Borel 3-color, so for convenience we assume that f is aperiodic. Fix a vanishing sequence $(A_n)_{n\in\omega}$ of fwd recurrent Borel markers, assuming for aesthetics that $A_0 = X$. Disjointify, putting $B_n = A_n \setminus A_{n+1}$, so that $X = \bigsqcup_n B_n$.

Claim 0: $\forall x \in X\ \exists^\infty n \in \omega\ \ f^\omega(x) \cap B_n \neq \emptyset$.

pf (C0): Let $m \in \omega$ be arbitrary, and fix some $y \in f^\omega(x) \cap A_m$. Fix $n \geq m$ least with $y \notin A_{n+1}$. Then $y \in f^\omega(x) \cap B_n$ as desired. ■(C0)

We shall use the sequence $(B_n)_{n\in\omega}$ to parametrize attempts at building a fwd recurrent Borel set whose complement is also Borel.

Towards that end, define for $\sigma \in 2^\omega$ the set

$$B_\sigma = \bigcup\{B_n : \sigma(n) = 1\}.$$

Define also a Borel set $C \subseteq X \times 2^\omega$ by

$$(x,\sigma) \in C \text{ iff } \exists^\infty m,n\ \ f^m(x) \in B_\sigma \wedge f^n(x) \notin B_\sigma$$

In other words, B_σ and $X \setminus B_\sigma$ both meet $f^\omega(x)$ infinitely often.

4 pf(Thm, cont.)

Claim 1: $\forall x \in X$, C_x is a comeager subset of 2^ω.

pf(C1): Fix $x \in X$ and $s \in 2^{<\omega}$. We want to find some $t \sqsupseteq s$ s.t. $\sigma \in N_t \Rightarrow B_\sigma$ and $X \setminus B_\sigma$ meet $f^\omega(x)$. By Claim 0, find $m \neq n \geq \mathrm{len}(s)$ such that B_m and B_n both meet $f^\omega(x)$. Extend s to any $t \in 2^{<\omega}$ with $t(m) = 1$ and $t(n) = 0$. Then for any $\sigma \in N_t$ we see

$$B_m \subseteq B_\sigma \quad \text{and} \quad B_n \subseteq X \setminus B_\sigma$$

ensuring that B_σ and $X \setminus B_\sigma$ meet $f^\omega(x)$. ∎(C0)

In other words,

$$\forall x \in X \ \forall^* \sigma \in 2^\omega \ (x, \sigma) \in C.$$

Kuratowski-Ulam then yields

$$\forall^* \sigma \in 2^\omega \ \forall^* x \in X \ (x, \sigma) \in C.$$

In particular, there is $\sigma \in 2^\omega$ such that the set

$$Y = C^\sigma = \{x \in X : |f^\omega(x) \cap B_\sigma| = |f^\omega(x) \cap (X \setminus B_\sigma)| = \aleph_0\}$$

is E_f-invariant and comeager. In particular, B_σ and $Y \setminus B_\sigma$ are both fwd recurrent for $f \restriction Y$. As discussed last time, this implies that $\chi_B(G_f \restriction Y) \leq 3$ as desired. ∎(Thm)

DST

Lecture 36

Hypersmoothness (of CBERs)

Def: $\mathbb{E}_1$ is the eq rel of eventual agreement on $\mathbb{R}^\omega$:

$$x \,\mathbb{E}_1\, y \text{ iff } \exists m \in \omega\, \forall n \geq m\ x(n) = y(n).$$

Thm (Dougherty-Jackson-Kechris, 1994):

Suppose that E is a CBER on X. TFAE:

$\boxed{\text{I}}$ E is hyperfinite

$\boxed{\text{II}}$ E is hypersmooth

$\boxed{\text{III}}$ $E \leq_B \mathbb{E}_1$

$\boxed{\text{IV}}$ $E \leq_B E_f$ for some Borel function f

$\boxed{\text{V}}$ $E \leq_B E_g$ for some (≤ 2)-to-1 Borel function g.

pf: $\boxed{\text{I}} \Rightarrow \boxed{\text{II}}$:

$\boxed{\text{II}} \Rightarrow \boxed{\text{III}}$: Fix increasing smooth E_n with $E = \bigcup_n E_n$.

Fix Borel reductions $\varphi_n : X \to \mathbb{R}$ from E_n to $\Delta_{\mathbb{R}}$.

Then the map $\Theta : X \longrightarrow \mathbb{R}^\omega$ defined by

$$\Theta(x) : n \longmapsto \varphi_n(x)$$

is a Borel reduction from E to $\mathbb{E}_1$.

② pf(Thm, cont):

$\boxed{\text{III}} \Rightarrow \boxed{\text{IV}}$: It suffices to show that $\mathbb{E}_1$ has the form E_f for some Borel $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$.

For example, the function that "zeroes out" the first nonzero entry does the job. I.e.,

$$f : \begin{cases} 0^n{}^\frown r^\frown z \mapsto 0^{n+1}{}^\frown z & \text{for } n\in\omega,\ r\in\mathbb{R}\setminus\{0\},\ z\in\mathbb{R}^\omega \\ 0^\omega \mapsto 0^\omega \end{cases}$$

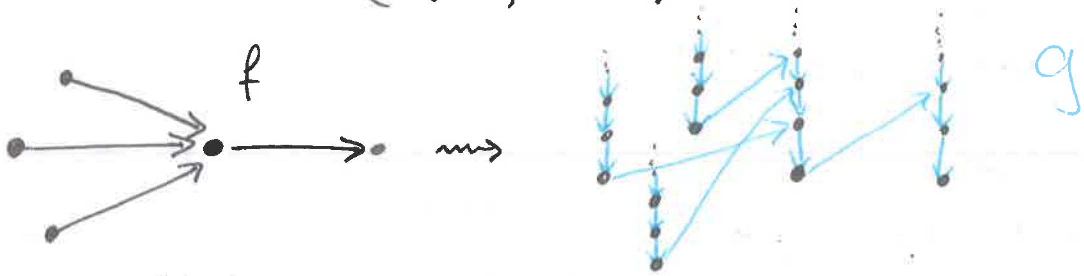
✓

$\boxed{\text{IV}} \Rightarrow \boxed{\text{V}}$: By replacing E with the forward f-saturation of the image of the reduction, we may assume that $E = E_f$ for notational convenience. As E is a CBER, we may fix a Borel edge ω-coloring

$$c : G_f \longrightarrow \omega.$$

We now define a Borel $g : X \times \omega \to X \times \omega$ via

$$g : (x,m) \longmapsto \begin{cases} (f(x),\ c(x, f(x))) & \text{if } m = 0 \\ (x,\ m-1) & \text{if } m > 0. \end{cases}$$



Observe that g is (≤ 2)-to-1, as if $g(x,m) = (y,n)$, either $(x,m) = (y, n+1)$ OR $(x,m) = (x,0)$ and x satisfies $c(x,y) = n$. Also, the map $x \mapsto (x,0)$ is a Borel reduction from E_f to E_g.

✓

③ pf (Thm, cont.):

$\boxed{V} \Rightarrow \boxed{I}$: Again, to simplify notation we shall assume that $E = E_g$ for some (≤ 2)-to-1 Borel function $g : X \to X$, and aim to establish hyperfiniteness of E. As the periodic part of g is smooth, WLOG we may assume that g is aperiodic.

By the Marker Lemma, fix a vanishing sequence $(A_m)_{m \in \omega}$ of fwd recurrent Borel markers, assuming for aesthetics that $A_0 = X$.

Def: ⓐ For $m, n \in \omega$ define Borel $s^m_n : X \to X$ by

$$s^m_n : x \longmapsto \begin{cases} g^i(x) & \text{where } i \leq n \text{ is least with } g^i(x) \in A_m, \\ x & \text{if no such } i \text{ exists} \end{cases}$$

In particular, $x \in A_m \Rightarrow \forall n \; s^m_n(x) = x = g^0(x)$.

ⓑ For $n \in \omega$ define Borel $w_n : X \to X$ by

$$w_n = s^n_n \circ s^{n-1}_n \circ \cdots \circ s^1_n \circ s^0_n$$

We view s^m_n as an attempt to step to A_m with a stride of length (at most) n, and w_n as the walk resulting from stepping to $A_0, A_1, \ldots, A_n$ in turn.

Claim 0: Each s^m_n is $(< 2^{n+1})$-to-1, and each w_n is $(< 2^{n^2+n})$-to-1 (maybe?)

pf (C0): Induction, using that g is (≤ 2)-to-1. ■(C0)

We may thus define FBERs F_n on X via

$$x F_n y \text{ iff } w_n(x) = w_n(y).$$

As $w_n(x) \, E_g \, x$, it follows that $F_n \subseteq E_g$.

④ pf (Thm, cont.): $\boxed{\text{V}} \Rightarrow \boxed{\text{I}}$ (cont.)

We shall see that $(F_n)_{n\in\omega}$ witnesses hyperfiniteness of E_g.

Claim 1: $F_n \subseteq F_{n+1}$

pf (C1): Suppose that $x \, F_n \, y$, and put $z = w_n(x) = w_n(y)$. Fix $m \leq n$ largest with $z \in A_m$ (certainly $z \in A_0 = X$) and observe that

$$z = s_n^n \circ \cdots \circ s_n^0(x) = s_n^m \circ \cdots \circ s_n^0(x)$$
$$= s_n^n \circ \cdots \circ s_n^0(y) = s_n^m \circ \cdots \circ s_n^0(y)$$

as $\forall i > m$ s_n^i does nothing (else we'd have $z \in A_i$). It follows that $s_{n+1}^m \circ \cdots \circ s_{n+1}^0(x) = z = s_{n+1}^m \circ \cdots \circ s_{n+1}^0(y)$ and thus $w_{n+1}(x) = w_{n+1}(y)$ as desired. ■(C1)

Claim 2: $E_g = \bigcup_n F_n$.

pf (C2): It suffices to show $E_g \subseteq \bigcup_n F_n$. Suppose that $x \, E_g \, y$. The disagreement between their forward orbits, $g^\omega(x) \Delta g^\omega(y)$, is finite, so fix large m so that A_m is disjoint from this set. Let $z \in A_m$ be the first element of A_m in $g^\omega(x)$, equivalently the first element of A_m in $g^\omega(y)$. Finally, pick $n \in \omega$ sufficiently large so that

$$z = g^i(x) = g^j(y) \quad \text{for } i, j < n.$$

It follows that

$$z = s_n^m \circ \cdots \circ s_n^0(x)$$
$$= s_n^m \circ \cdots \circ s_n^0(y)$$

and thus that $w_n(x) = w_n(y)$ as desired. ■(C2)

So $(F_n)_{n\in\omega}$ witnesses hyperfiniteness of E_g. ✓

■(Thm)

DST

Lecture 37

The essential uncountability of $\mathbb{E}_1$

<u>Recall</u>: $\mathbb{E}_1$ is the eq. rel. of eventual agreement on $\mathbb{R}^\omega$.

Today we will show that $\mathbb{E}_1$ is NOT Borel reducible to any CBER. But first, we need a basic fact that, inexplicably, we haven't yet seen.

<u>Prop</u>: Suppose that X, Y are Polish and $\varphi: X \to Y$ is Borel (or just BP-measurable). Then there is a comeager set $C \subseteq X$ so that $\varphi \upharpoonright C$ is continuous.

<u>pf</u>: For each open $V \subseteq Y$, we know that $\varphi^{-1}(V)$ is a BP subset of X. This means we may write $\varphi^{-1}(V) = \mathcal{U}_V \triangle M_V$ for some open $\mathcal{U}_V \subseteq X$ and some meager $M_V \subseteq X$.

Now fix a countable base $\mathcal{B}$ for the topology on Y, and put $C = X \setminus \bigcup\{M_V : V \in \mathcal{B}\}$. Certainly C is comeager, and $\varphi \upharpoonright C$ is continuous as for all $V \in \mathcal{B}$ we have $\varphi^{-1}(V) \cap C = \mathcal{U}_V \cap C$. ∎(Prop)

<u>Remark</u>: Of course, the previous proposition holds for a much broader class of topological spaces.

Thm (Kechris–Louveau, 1997):
Suppose that F is a CBER on X, and that $\varphi : \mathbb{R}^\omega \to X$ is a Borel homomorphism from $\mathbb{E}_1$ to F. Then there are $\alpha, \beta \in \mathbb{R}^\omega$ s.t.

- $\forall m \in \omega \quad \alpha(m) \neq \beta(m)$
- $\varphi(\alpha) = \varphi(\beta)$.

In particular, φ is not a reduction.

pf: With loss of generality, we assume that X is Polish and that φ is continuous. This is true "off a meager set" by our previous Proposition, and afterwards we discuss how to tweak the argument to avoid this meager set.

We shall recursively construct $a_n, b_n \in \mathbb{R}^n$ s.t.

- $a_n \sqsubseteq a_{n+1}, \quad b_n \sqsubseteq b_{n+1}$
- $\forall m < n \quad a_n(m) \neq b_n(m)$
- $\exists^* \gamma \in \mathbb{R}^\omega \quad \varphi(a_n{}^\frown \gamma) = \varphi(b_n{}^\frown \gamma)$.

As usual, $\exists^* \gamma \in \mathbb{R}^\omega \; P(\gamma)$ abbreviates

"$\{\gamma \in \mathbb{R}^\omega : P(\gamma)\}$ is nonmeager"

and dually, $\forall^* \gamma \in \mathbb{R}^\omega \; Q(\gamma)$ abbreviates

"$\{\gamma \in \mathbb{R}^\omega : Q(\gamma)\}$ is comeager."

pf (Thm, cont.)

Stage 0: Take $a_0 = b_0 = \emptyset$. ☺

Stage n+1: We have $a_n, b_n \in \mathbb{R}^n$ and, by localization, may fix basic open $W' \subseteq \mathbb{R}^n$ such that

$$\forall^* \gamma \in W' \ \varphi(a_n {}^\frown \gamma) = \varphi(b_n {}^\frown \gamma).$$

For fixed $\gamma \in W'$, we consider the map

$$f_\gamma : \mathbb{R} \longrightarrow X$$
$$r \longmapsto \varphi(a_n {}^\frown r {}^\frown \gamma).$$

Then $f_\gamma[\mathbb{R}]$ is contained in a single (countable) F-class, as φ is a hom from $\mathbb{E}_1$ to F. Thus there is a nonmeager BP set on which f_γ is constant. Localization yields basic open $\mathcal{U}_\gamma \subseteq \mathbb{R}$ s.t. $\forall^* r \in \mathcal{U}_\gamma \ \forall^* s \in \mathcal{U}_\gamma \ f_\gamma(r) = f_\gamma(s)$.

Now, there are only countably many options for $\mathcal{U}_\gamma$, so some $\mathcal{U}$ works for a nonmeager BP set of $\gamma \in W'$. Localize again, fixing $W \subseteq W'$ s.t.

$$\forall^* \gamma \in W \ \forall^* r \in \mathcal{U} \ \forall^* s \in \mathcal{U} \ \varphi(a_n {}^\frown r {}^\frown \gamma) = \varphi(b_n {}^\frown s {}^\frown \gamma).$$

Kuratowski-Ulam yields

$$\forall^* r \in \mathcal{U} \ \forall^* s \in \mathcal{U} \ \forall^* \gamma \in W \ \varphi(a_n {}^\frown r {}^\frown \gamma) = \varphi(b_n {}^\frown s {}^\frown \gamma).$$

Pick distinct r, s that work as above, and declare

$$a_{n+1} = a_n {}^\frown r$$
$$b_{n+1} = b_n {}^\frown s.$$

This completes the recursive construction.

④ pf (Thm, cont.):

We now put $\alpha = \bigcup_n a_n$ so $\forall m \in \omega\ \alpha(m) \neq \beta(m)$.
$\beta = \bigcup_n b_n$

We also fix $\forall n \in \omega$ some $\gamma_n \in \mathbb{R}^\omega$ such that

$$\varphi(a_n{}^\frown\gamma_n) = \varphi(b_n{}^\frown\gamma_n)$$

as guaranteed by our construction.

Observe that
$$\lim_n a_n{}^\frown\gamma_n = \alpha$$
$$\lim_n b_n{}^\frown\gamma_n = \beta.$$

Finally, continuity of φ implies that
$$\varphi(\alpha) = \varphi(\beta)$$
as desired. ☺ (Thm)

OK, let's discuss how to handle Borel φ. The Proposition yields comeager C so that $\varphi \upharpoonright C$ is continuous. The only place that we used continuity of φ in the above argument is in the very last step. Thus, it is enough to ensure that all of
$$a_n{}^\frown\gamma_n,\ b_n{}^\frown\gamma_n,\ \alpha,\ \beta$$
end up in C.

To accomplish this, fix open dense $O_n \subseteq \mathbb{R}^\omega$ with $\bigcap_n O_n \subseteq C$, and as we recursively build a_n and b_n simultaneously build shrinking open $A_n, B_n \subseteq O_n$ in which the remainder of the construction will be performed. ■ (Thm)

①

DST

Lecture 38

The dark side (after Greg Hjorth, 1963-2011)

Recall: $\mathbb{E}_1$ is the eq rel of eventual agreement on $\mathbb{R}^\omega$.

Today's goal: $\mathbb{E}_1$ is not Borel reducible to any orbit equivalence relation of a Borel action of a Polish group.

Def: ⓐ A *topological group* is a group whose underlying set is equipped with a topology rendering the group operation and inversion continuous.

ⓑ A *Polish group* is a Polish topological group.

Example: $(\mathbb{R}^\omega; +)$ equipped with the ptwise conv. top.

Notation: For $n\in\omega$, put $H_n = \{\alpha\in\mathbb{R}^\omega : \forall i\geq n\ \alpha(i)=0\}$, so $H_n \leq \mathbb{R}^\omega$ and $H_n \cong \mathbb{R}^n$.

Put $\mathbb{R}^{<\omega} = \bigcup_n H_n$ (sometimes called c_{00} ☹)

Observation: $\mathbb{R}^{<\omega}$ is a "standard Borel group" acting on $\mathbb{R}^\omega$ with orbit equivalence relation $\mathbb{E}_1$. This seems like a contradiction with today's goal, but it's not.

② <u>Thm</u> (Folklore): $\mathbb{R}^{<\omega}$ is not Polishable, i.e., there is no Polish group top compatible w/its Borel structure.

<u>pf</u> (Sketch): Suppose we had such a Polish topology. Since $\mathbb{R}^{<\omega} = \bigcup_n H_n$, we could fix $n \in \omega$ with H_n BP nonmeager. A Pettis/Steinhaus argument implies that H_n has non-$\emptyset$ interior, which in turn implies that H_n is clopen. So, by separability, $\mathbb{R}^{<\omega}/H_n$ must be countable, contradicting $\mathbb{R}^{<\omega}/H_n \cong \mathbb{R}^{<\omega}$. ■ (Sketch)

<u>Thm</u> (Kechris-Louveau, 1997):

Suppose that G is a Polish group acting in a Borel fashion on std Borel X with orbit equivalence relation F. Then $\mathbb{E}_1 \not\leq_B F$.

<u>pf</u> (Hjorth, sketch):

Fix a complete metric d_G on G compatible with its Polish topology. By an important theorem of Becker-Kechris, we may fix a Polish topology on X rendering the action $G \curvearrowright X$ continuous.

Now, suppose that $\varphi: \mathbb{R}^\omega \to X$ is a Borel homomorphism from $\mathbb{E}_1$ to F. We aim to show that it cannot be a reduction, i.e., some pair of $\mathbb{E}_1$-unrelated points map to the same F-class.

pf(Thm, cont.):

Let us abbreviate "$W \subseteq G$ is open and $1_G \in W$" by the convenient notation $W \subseteq_o G$. Since φ is a hom from $\mathbb{E}_1$ to F, we know for $n \in \omega$

$$\forall \alpha \in \mathbb{R}^\omega \ \forall h \in H_n \ \exists g \in G \ (\varphi(h+\alpha) = g \cdot \varphi(\alpha)).$$

We would like the dependence of g upon h to be continuous, which we accomplish mod meager by an orbit continuity lemma:

Lemma: There is a comeager $C \subseteq \mathbb{R}^\omega$ satisfying:

- $\varphi \restriction C$ is continuous
- $\forall n \in \omega \ \forall \alpha \in C \ \forall W \subseteq_o G \ \exists V \subseteq_o H_n$ so that

$$\forall^* h \in V \ \exists g \in W \ (\varphi(h+\alpha) = g \cdot \varphi(\alpha)).$$

pf($\mathcal{L}$): $\boxed{\ddot\smile}$ ($\mathcal{L}$)

For notational convenience, we assume $C = \mathbb{R}^\omega$ acknowledging that a meager set must be sidestepped as usual.

We fix a summable sequence $(\varepsilon_n)_{n\in\omega}$ of positive reals, and recursively construct:

$$\text{positive } h_n \in H_n \ [\text{i.e., } \forall i<n \ h_n(i) > 0]$$
$$\alpha_n \in \mathbb{R}^\omega$$
$$g_n \in G$$

satisfying:

- $\alpha_n = \sum_{m \leq n} h_m$
- $\forall i<n \ \ h_n(i) < \varepsilon_n$
- $d_G(g_{n+1}, g_n) < \varepsilon_n$
- $g_n \cdot \varphi(\vec{0}) = \varphi(\alpha_n)$

pf (Thm, cont.)

Stage 0: Take $h_0 = \alpha_0 = \vec{0}$, $g_0 = 1_G$.

Stage n+1: Given h_n, α_n, g_n, consider
$$W = \{g \in G : d_G(gg_n, g_n) < \varepsilon_n\} \subseteq_0 G.$$
The orbit continuity lemma grants $V \subseteq_0 H_{n+1}$ s.t.
$$\forall^* h \in V \ \exists g \in W \ (\varphi(h + \alpha_n) = g \cdot \varphi(\alpha_n))$$
Pick small positive $h_{n+1} \in V$ and corresponding $g \in W$ as above. Putting $\alpha_{n+1} = h_{n+1} + \alpha_n$ and $g_{n+1} = g g_n$, we see
$$\varphi(\alpha_{n+1}) = g \cdot \varphi(\alpha_n) = g \cdot g_n \cdot \varphi(\vec{0}) = g_{n+1} \cdot \varphi(\vec{0})$$
as desired.

The construction is complete. Put $\alpha = \lim_n \alpha_n$
$g = \lim_n g_n$.

Claim 0: $\vec{0} \not{E}_1 \alpha$

pf (C0): $\forall m \in \omega$ $\alpha(m) > 0$ by positivity of h_{m+1}. ∎ (C0)

Claim 1: $\varphi(\vec{0}) \ F \ \varphi(\alpha)$.

pf (C1): We compute
$$\begin{aligned} g \cdot \varphi(\vec{0}) &= \lim_n g_n \cdot \varphi(\vec{0}) && [\text{cont. of } G \curvearrowright X] \\ &= \lim_n \varphi(\alpha_n) && [\text{construction}] \\ &= \varphi(\lim_n \alpha_n) && [\text{cont. of } \varphi] \\ &= \varphi(\alpha) \end{aligned}$$
∎ (C1)

The claims preclude φ from being a reduction.

∎ (Thm, sketch)