

PROBLEMS ON h^* -VECTORS FOR MATROIDS

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ABSTRACT. Matroid Ehrhart theory marries two branches: (1) Ehrhart theory of lattice polytopes (a well-established and still-active field) and (2) K -theoretic positivity for hyperplane arrangements and matroids (a new development [EL26, EFL, BF, Liu]). We collect problems about h^* -vectors for matroids, as introduced in [EFL26], where many of the problems are motivated by analogous results/questions about h^* -vectors of lattice polytopes surveyed in [FH24].

For any suggestions/comments/questions, or if you have made a substantial progress on a problem, please consider letting us know at ceur@cmu.edu.

NOTATION AND CONVENTIONS

We follow the notations and conventions as given in [EFL26] and [FH24]. A pair (M, P) consists of M a loopless matroid on $[n]$ and $P \subseteq \mathbb{R}^{[n]}$ a lattice generalized permutohedron (GP). We write

$$d = \deg(\ell \mapsto \chi(M, \mathcal{L}_P^\ell)), \quad h^*(M, \mathcal{L}_P) = h^*(M, P) = (h_0^*, \dots, h_d^*), \quad s = \max\{i : h_i^* \neq 0\},$$

where $\chi(M, -)$ is the sheaf Euler characteristic on the matroid K -ring and $\mathcal{L}_P$ is the (nef) line bundle associated with P (see [LLPP24] and [EL26]).

Code for computing $h^*(M, P)$ can be found at <https://github.com/chrisweur/hStarMatroid>.

Caution: When P is a GP, so is $-P$. This can cause lots of confusion along with the min vs. max convention issue. While the max convention is often used for associating a GP with a submodular function, we follow the min convention to match the toric geometry conventions of [Ful93, CLS11]. The result is that $h^*(M, \mathcal{L}_\Delta) = (1, 0, \dots, 0)$ where Δ is the standard $(n-1)$ -simplex $\text{conv}\{e_i : i \in [n]\}$ in $\mathbb{R}^n$; that is, geometrically $\mathcal{L}_\Delta$ is the $\mathcal{O}(1)$ on the projective space pulled back to the wonderful variety via the blow-up. See also [BEST23, Section 2.7 & Remark III.1].

PROBLEMS

Problem 1. Show that, for a fixed matroid M and an integer $\ell > 0$, there are finitely many GPs P (up to defining the same element in $A^1(M)$) satisfying $h_d^*(M, P) = \ell$. Further, give bounds for the number of such P in terms of invariants like “volume.”

Motivated by: For lattice polytopes, the last h^* -vector entry is the number of interior lattice points. Lagarias and Ziegler [LZ91] bounded the number of lattice polytopes in $\mathbb{R}^n$ (up to unimodular equivalence) with a given number of interior lattice points.

Status: First part solved by Ruizhen Liu and the author, in a forthcoming work. The further task of giving bounds remains to be done.

Problem 2. Show that if GPs P and Q are full-dimensional (or more generally, have the same induced partition), then $P \subseteq Q$ implies $h^*(M, P) \leq h^*(M, Q)$ coefficientwise.

Motivated by: Known as Stanley monotonicity [Sta93, JS18]. For the sum of the entries, monotonicity is given in [EL26, Proposition 5.13].

Status: Open. In the special case where P is a face of a subdivision of Q into GPs, solved by Sharon Robins and the author, in a forthcoming work. There are however examples of $P \subseteq Q$ where no GP-subdivision of Q has P as a face [Kas, Example 53].

Problem 3. Show that if $M \rightarrow M'$ is a rank-preserving weak map of matroids, and P is a full-dimensional GP, then $h^*(M, P) \geq h^*(M', P)$ coefficientwise.

Motivated by: The (unsigned) characteristic polynomial has such a monotonicity property, as well as a few other well-known matroid invariants [Luc75, FS24].

Status: Open. The methods developed by Robins and the author for the previous problem likely apply to solve the special case where M is the uniform matroid.

Problem 4. Show that $h_0^* + h_1^* + \cdots + h_i^* \leq h_s^* + h_{s-1}^* + \cdots + h_{s-i}^*$ for all $0 \leq i \leq s$.

Motivated by: These are known as Stanley's inequalities, which hold for any (semistandard) graded Cohen–Macaulay algebra domains [Sta91, Theorem 2.1].

Status: Immediate when M is realizable, but unclear for arbitrary matroids.

Problem 5. Show that $h_1^* \leq h_i^*$ for all $1 \leq i \leq d-1$ when $s = d$. More strongly, show that for $a \geq 0$ and $b \geq 1$ with $a + b < s$, one has $h_1^* + \cdots + h_b^* \leq h_{a+1}^* + \cdots + h_{a+b}^*$.

Motivated by: The first inequality is known as Hibi's lower bound theorem [Hib94, Theorem 1.1] for lattice polytopes (not necessarily IDP). The second holds for all standard graded Cohen–Macaulay algebra domains over characteristic zero [Sta91, Proposition 3.4].

Status: Immediate for M realizable over characteristic zero, but unclear beyond those.

Problem 6. Is there a linear inequality involving h_1^* and h_d^* for $h^*(M, P)$ (and possibly involving some invariants of M) that is analogous to the following known (easy) inequality for lattice polytopes: $h_1^* \geq h_d^*$?

Motivated by: [FH24] calls $h_1^* \geq h_d^*$ the “most elementary inequality” (via $h_1^* = |P \cap \mathbb{Z}^n| - (d+1)$ and $h_d^* = |P^\circ \cap \mathbb{Z}^n|$), but it fails for $h^*(M, P)$ ($\because$ when P is the negated standard simplex, $h^*(M, P)$ is $T_M(t, 0)/t$ read backwards).

Status: Open. The inequality as written, $h_1^* \geq h_d^*$, is generally false, except in the following special case: when P is “simplicially positive” (i.e. P is a nonnegative Minkowski sum of faces of the standard simplex), ChatGPT 5.5 Pro produced a proof of a stronger set of inequalities.

Problem 7. Is there a linear inequality that is analogous to the following known inequality for lattice polytopes: $h_2^* + \cdots + h_{i+1}^* \geq h_{d-1}^* + \cdots + h_{d-i}^*$ for all $1 \leq i \leq \lfloor \frac{d}{2} \rfloor - 1$?

Motivated by: Known for lattice polytopes by Hibi (in a remark) and Stapledon [Sta09, Sta16].

Status: Open. The inequality as written is generally false.

Problem 8. Is $h^*(M, P)$ always strongly flawless, i.e. the following two inequalities are always satisfied: $h_0^* \leq h_1^* \leq \cdots \leq h_{\lfloor \frac{s}{2} \rfloor}^*$ and $h_i^* \leq h_{s-i}^*$ for all $0 \leq i \leq s$? (Is either one always satisfied?)

Motivated by: When P is the negated standard simplex, in which case h^* is the h -vector of the broken-circuit complex of M , this is the main result of [JKL18].

Status: Open.

Problem 9. (When) is $h^*(M, P)$ unimodal? Log-concave? Real-rooted?

Motivated by: A long-standing question in Ehrhart theory asks whether the h^* -vector of an IDP polytope is unimodal. A non-log-concave example was recently found [HKN].

Status: Open. Non-real-rooted examples abound, but non-examples for the other two properties seem difficult to find (if they exist).

Problem 10. Provide a combinatorial interpretation for h^* of hypersimplices, i.e. $h^*(M, P(U_{k,n}))$ where $P(M)$ denotes the base polytope of a matroid M .

Motivated by: The sum of the h^* -vector in these cases is given by [BST23]. For the first and last hypersimplices, one obtains $(1, 0, \dots, 0)$ and the coefficients of the Tutte evaluation $T_M(t, 0)$.

Status: Resolved by ChatGPT 5.6 Sol Pro after three guided prompts (7/25/2026), forthcoming work to be made public soon.

Problem 11. Provide explicit formulas for $h^*(M, P)$ when M is a (loopless) Schubert matroid and P is the base polytope of a Schubert matroid.

Motivated by: Because $\chi(M, \mathcal{L}_P)$ is “bi-valuative” in M and P , the h^* for these cases determines the general case [DF10, EHL23]. May turn out computationally useful as well.

Status: Resolved by ChatGPT 5.6 Sol Pro with a single guided prompt (7/25/2026), forthcoming work to be made public soon.

Problem 12. Characterize the pairs for which $h^*(M, P)$ is palindromic. In such cases, is there necessarily a Gorenstein graded ring whose Hilbert series matches $h^*(M, P)$?

Motivated by: When M is boolean and P is a matroid base polytope, this was done in [LM23]. The second part of the problem is always yes in the polytope case, since Cohen–Macaulay domains with a symmetric Hilbert series numerator are Gorenstein [Sta78, Theorem 4.4].

Status: Open. The second part is immediately yes for realizable M .

Problem 13. For palindromic $h^*(M, P)$, when is it further γ -positive?

Motivated by: Gal’s conjecture asserts that a flag simplicial sphere has a γ -positive h -polynomial. An analogue for h^* -polynomials of Gorenstein polytopes appears as [FH24, Conjecture 3.28]. A GP has a regular unimodular triangulation [BL25], but not necessarily a flag regular unimodular triangulation [BCL⁺, LFMR].

Status: Open.

Problem 14. “Magic positivity”: For which (M, P) does the polynomial $x \mapsto \chi(M, \mathcal{L}_P^x)$ expand with nonnegative coefficients in the basis $\{x^i(1+x)^{d-i}\}$?

Motivated by: Magic positivity implies real-rootedness, and was used to great effect for certain families of lattice polytopes and simplicial complexes; see [FH24, Section 4.5] for a brief overview.

Status: Open.

Problem 15. Is the Ehrhart series log-concave, i.e. is the sequence $(\chi(M, \mathcal{L}_P^k))_{k=0,1,\dots}$ log-concave?

Motivated by: [FH24, Conjecture 5.7] conjectures that IDP lattice polytopes have log-concave Ehrhart series.

Status: Open, but the first step is easy. Degree-1 generation gives surjective multiplication $S_n \otimes S_m \rightarrow S_{n+m}$, hence submultiplicativity $\chi(M, \mathcal{L}_P^n)\chi(M, \mathcal{L}_P^m) \geq \chi(M, \mathcal{L}_P^{n+m})$, so the $n = m = 1$ case $\chi(M, \mathcal{L}_P)^2 \geq \chi(M, \mathcal{L}_P^2)$ holds.

Problem 16. Is the *interior* Ehrhart series log-concave, i.e. is the sequence $(|\chi(M, \mathcal{L}_P^{-k})|)_{k=1,2,\dots}$ log-concave? Is it at least always log-concave at $k = 2$, i.e. $|\chi(M, \mathcal{L}_P^{-2})|^2 \geq |\chi(M, \mathcal{L}_P^{-1})||\chi(M, \mathcal{L}_P^{-3})|$?

Motivated by: See [FH24, Section 5.2] for the connection to K -polynomials, which behave particularly well in the simplicially positive case [CCRMM, EL26].

Status: Open.

Problem 17. For a given linear subspace $L \subseteq \mathbb{k}^n$, can one identify the class of lattice polytopes Q for which $f_Q(W_L)$ is Cohen–Macaulay (CM)? And variations thereof: CM for all linear subspaces $L \subseteq \mathbb{k}^n$ realizing a fixed matroid M , CM for all linear subspaces $L \subseteq \mathbb{k}^n$ for either a fixed field $\mathbb{k}$ or all fields $\mathbb{k}$, etc.

Motivated by: A main result of [EFL] states that $f_Q(W_L)$ is always CM if Q is a GP.

Status: Open. See [EFL26, Remark 7] for some examples.

Acknowledgements. We thank Felipe Rincón and Daniel Hwang for suggesting Problem 11, and Jesús De Loera for suggesting Problem 17. We thank Matt Larson for many helpful conversations, as well as suggesting Problems 5 and 8. Problems 2, 4 and 5 appear as Conjectures 4.6.4, 4.6.6 and 4.6.7 (respectively) in [Lar24]. We thank Junaid Hasan, Sharon Robins, Jay Yang, Josephine Yu, and Michael Zeng for testing many of the problems via AI during the “AG & ML” workshop at ICARM (June 2026).

Acknowledgement of AI usage. Claude Opus 4.8 Max was used for the first round of generating the problems as well as a quick sanity check for each of the problems, after which the author curated, rephrased, modified, and added problems along with further discussion for each problem. Many of these problems were attempted by ChatGPT 5.5 Pro during the “AG & ML” workshop at ICARM (June 2026), with no substantial progress except for a special case of Problem 6.

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