A Unified Framework for Utility Maximization Problems: An Orlicz Space Approach

Sara Biagini[*] and Marco Frittelli[§],
[(*)] University of Perugia, [(§)] University of Firenze

Extended abstract

In the most general semimartingale model for the underlying process X, the problem we address takes the following form:

$$\sup_{H \in \mathcal{H}} E[u(x + (H \cdot X)_T)] \tag{1}$$

where

- u is the utility function of the agent, which is assumed to be concave, strictly increasing and differentiable over its proper domain;
- x is the initial endowment of the agent and $T \in (0, \infty]$ is the time horizon;
- $\mathcal{H}$ is a proper class of admissible $\mathbb{R}^d$-valued predictable processes, which represent the allowed trading strategies;
- $(H \cdot X)_T$ is the terminal gain of the investor when following a strategy H.

Expected utility maximization in continuous-time stochastic incomplete markets is a very well known problem that received a great impulse in the middle of the eighties when the "duality approach" to the resolution was first employed (Pliska 1986). Here we consider the literature that discusses this problem in the general context of semimartingale price processes and general classes of utility functions satisfying Inada and growth type conditions.

Up to now, the current literature is somehow split in two main branches, which rely on two different applications of the duality:

(A) The first case (see e.g. Cvitanic- Schachermayer-Wang (2001) and Kramkov-Schachermayer (1999)) is when the proper domain of u is $\mathbb{R}_+$ (i.e. log-like utility functions) and X is a general $\mathbb{R}^d$-valued semimartingale.

(B) In the second case the utility functions have $\mathbb{R}$ as the proper domain (exponential-like utility functions):

  - When X is locally bounded the problem is addressed by Schachermayer (2001): The set $\mathcal{H}$ of strategies here employed is the classical set $\mathcal{H}^1$ of strategies with uniformly bounded from below wealth.
  - When X is a general, not necessarily locally bounded, $\mathbb{R}^d$-valued semimartingale the problem is dealt in Biagini-Frittelli (2005) and is based on a careful analysis of the proper set of strategies $\mathcal{H}$ that are allowed in the trading.

Indeed, the traditional set $\mathcal{H}^1$ of strategies may reduce to the null strategy when X is not locally bounded (just to fix the ideas, think of such an X as a Compound Poisson with unbounded jump size). So the maximization problem on this set turns out to be trivial.

To model the situation in which the investor is willing to take more risk to really increase his/her expected utility in a very risky market, in Biagini-Frittelli (2005) we enlarged the set of allowed strategies by admitting losses bounded from below by $-cW$, where W is a positive random variable, possibly unbounded from above. We defined the set $\mathcal{H}^W$ of W-admissible strategies by:

$$\mathcal{H}^W = \{H \in L(X) \mid (H \cdot X)_t \geq -cW \quad \forall t \leq T, \text{ for some } c > 0\}.$$

The stochastic integrals formed with these strategies enjoy good mathematical properties when it is assumed that the random variable W that controls the losses is:
(i) *compatible* with the preferences of the agent, in the sense that:

$$\forall \alpha > 0 \ E[u(x - \alpha W)] > -\infty; \tag{2}$$

(ii) *suitable* with the process X, i.e. there exists, for each $i = 1, ..., d$, a predictable X^i-integrable process H^i such that

$$P(\{\omega \mid \exists t \geq 0 \, H^i_t(\omega) = 0\}) = 0$$

and

$$-W \leq (H^i \cdot X^i)_t \leq W, \text{ for all } t \in [0, T], \ P - a.s.$$

In Biagini-Frittelli (2005) we formulated and analyzed by duality methods the utility maximization problem on the new domain $\mathcal{H}^W$. We showed that:

(a) For all loss variables W that are compatible and suitable, the optimal value on the class $\mathcal{H}^W$ coincides with the optimal value of the maximization problem over a larger domain K_Φ. The class K_Φ doesn't depend on the single W, but it depends on the utility function u through its conjugate function Φ;

(b) The optimal solution f_x exists in K_Φ, it can be represented as a stochastic integral $f_x = (H_x \cdot X)_T$ and the optimal wealth process $(H_x \cdot X)$ is a uniformly integrable martingale under the minimax measure and a supermartingale under each $\sigma-$martingale measures with finite generalized entropy (Biagini-Frittelli (2004));

(c) In general $H_x \notin \mathcal{H}^W$, so that the enlargement of the domain of the primal problem (from $\{(H \cdot X)_T \mid H \in \mathcal{H}^W\}$ to K_Φ) is necessary to catch the optimal solution.

The compatibility condition (2) guarantees that the infimum in the dual problem is attained by a *true* probability measure - compare with item 2 below. However, this condition may not be satisfied in some interesting cases (when X is an infinite activity Lévy process). The following example shows that this can happen even in a simple model.

Example 1 *Consider a single period market model with $X_0 = 1$ and trivial initial σ-algebra $\mathcal{F}_0$. Let $(\Omega, \mathcal{F}_1, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \frac{1}{2}e^{-|x|}dx)$ and let $X_1 = x$. Then $X = (X_0, X_1)$ is a semimartingale and it is obviously not locally bounded. A suitable W is $1 + |X_1|$, so that $\mathcal{H}^W = \mathbb{R}$. Suppose that the agent has zero initial endowment and $u(x) = -e^{-x}$. Then:*

$$E[u(-\alpha W)] > -\infty \text{ only if } \alpha \leq 1.$$

Hence (2) is not satisfied, but the weaker condition (3) holds true.

In the present paper:

1. We extend the above-mentioned results of Biagini-Frittelli (2005) by adopting the weaker compatibility condition:

$$\exists \alpha > 0 : E[u(x - \alpha W)] > -\infty, \tag{3}$$

   that allows considering more general market models.

2. We show that, in general, the optimal solution of the dual problem will have a singular component.

3. We prove that a duality relation holds true and we show the existence of the optimal solution to the primal problem.

The other main contribution of the present paper is on a more general level. We believe that there aren't good reasons for treating the problem (1) separately - for the two cases (A) and (B) - as it has been done up till now.

These two apparently different branches (A) and (B) can be seen as particular cases of a single, unified framework. In this paper the definitions of admissible trading strategies, the domains of the primal and dual optimization problems are the same for both cases. Moreover, the proofs of the main results are all formulated in the unified framework. Under the assumptions taken in (A) or in (B) we then deduce their results as corollaries of our theorems.

Following the ideas contained in Biagini 2005, we show that an elegant way to present this unified approach is to embed the utility maximization problem in the theory of Orlicz spaces.

References

[1] Biagini S. and M. Frittelli (2005) "Utility Maximization in Incomplete Markets for Unbounded Processes", Finance Stoc. **9/4**, pp 493-517**.**

[2] Biagini S. and M. Frittelli (2004) "The supermartingale property of the optimal portfolio process for general semimartingale", Technical Repor n. 9, Dept. "Matematica per le Decisioni", Università di Firenze, submitted.

[3] BIAGINI S. (2005) "An Orlicz Spaces Duality for Utility Maximization in Incomplete Markets", Proceedings of the 5th Ascona Seminar on Stoch. Analysis, Random Fields and Appl., Birkhauser, forthcoming.

[4] CVITANIC J., W. SCHACHERMAYER and H. WANG (2001): "Utility maximization in incomplete markets with random endowment", *Finance Stoc.* **5/2**, pp 259-272.

[5] KRAMKOV D. AND W. SCHACHERMAYER (1999): "The asymptotic elasticity of utility function and optimal investment in incomplete markets", *Ann. Appl. Prob.* **9/3**, pp 904-950.

[6] PLISKA S.R. (1986): "A stochastic calculus model of continuous trading: optimal portfolios", *Math. Oper. Res.*, **11/2**, pp 371-382.

[7] SCHACHERMAYER W. (2001): "Optimal investment in incomplete markets when wealth may become negative", *Ann. Appl. Prob.* **11/3**, pp 694-734.