Area

Varsity Practice 4/11/21

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1 Notation

If P is a polygon, then $[P]$ denotes its area. E.g. the area of triangle ABC is denoted $[ABC]$.

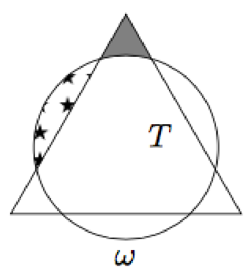
2 Warm-Up

1. (HMMT 2012) Let rectangle $ABCD$ have lengths $AB = 20$ and $BC = 12$. Extend ray BC to Z such that $CZ = 18$. Let E be the point in the interior of $ABCD$ such that the perpendicular distance from E to AB is 6 and the perpendicular distance from E to AD is 6. Let line EZ intersect $\overline{AB}$ at X and $\overline{CD}$ at Y. Find the area of quadrilateral $AXYD$.

2. A right triangle has legs of length 2 and 9. Find the length of the altitude to the hypotenuse *without using similar triangles.*

3 Problems

1. (HMMT 2016) In the below picture, T is an equilateral triangle with a side length of 5 and ω is a circle with a radius of 2. The triangle and the circle have the same center. Let X be the area of the shaded region, and let Y be the area of the starred region. What is $X-Y$?



2. (RMT 2009) $ABCD$ is a rhombus and point E is the intersection of AC and BD. Point F lies on AD such that $EF \perp FD$. Given that $EF = 2$ and $FD = 1$, find $[ABCD]$.

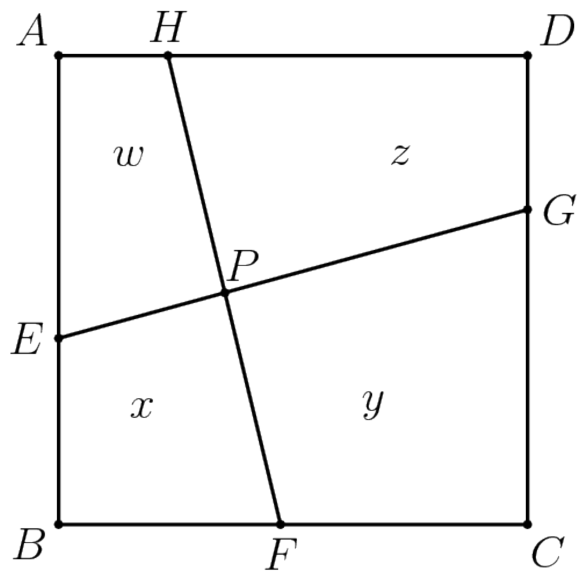
3. Let $ABCD$ be a convex quadrilateral and let E be the intersection point of the diagonals. If $[ABE] = 28$, $[BCE] = 40$, and $[CDE] = 75$, compute $[ADE]$.

4. (cf. RMT 2012) Let $ABCD$ be a rectangle with area 2016. There exist points E on AB and F on CD such that $DE = EF = FB$. Diagonal AC intersects DE at X and EF at Y. Compute $[EXY]$.

5. (ARML 2002) Let P be a point on side $\overline{ED}$ of regular hexagon $ABCDEF$ such that $\frac{EP}{PD} = \frac{3}{5}$. The line $\overline{CD}$ meets lines $\overline{AB}$ and $\overline{AP}$ at M and N respectively. Compute $\frac{[AMN]}{[ABCDEF]}$.

6. (a) Given a triangle ABC, describe the set of points D such that $[ABD] = [ACD]$.
   (b) Given a quadrilateral $ABCD$, describe the set of points E such that $[ABE] = [CDE]$.

7. (AIME 2014) On square $ABCD$, points E, F, G, and H lie on sides $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overline{DA}$, respectively, so that $\overline{EG} \perp \overline{FH}$ and $EG = FH = 34$. Segments $\overline{EG}$ and $\overline{FH}$ intersect at a point P, and the areas of the quadrilaterals $AEPH, BFPE, CGPF$, and $DHPG$ are in the ratio $269 : 275 : 405 : 411$. Find the area of square $ABCD$.



4 Challenge Problems

1. Let ABC be a triangle whose inscribed circle has radius r and circumscribed circle has radius R.
   (a) Prove that the maximal possible value of $\frac{[ABC]}{R^2}$ is attained when ABC is an equilateral triangle. *Hint: consider R and BC to be fixed, and show that if $[ABC]$ is maximal then $AB = AC$.*
   (b) Prove that the minimal possible value of $\frac{[ABC]}{r^2}$ is attained when ABC is an equilateral triangle.
   (c) Prove that the minimal possible value of $\frac{R}{r}$ is attained by an equilateral triangle.

2. (Putnam 2016) Suppose that S is a finite set of points in the plane such that the area of $\triangle ABC$ is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.