Set Theory 1

Varsity Practice 2/7/21
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1 Background

Sets

- Intersection: $X \cap Y = \{a |\ a \in X \wedge a \in Y\}$
- Union: $X \cup Y = \{a |\ a \in X \vee a \in Y\}$
- Set difference: $X \setminus Y = \{a \in X |\ a \notin Y\}$
- Symmetric difference: $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$
- Power set: $\mathscr{P}(X)$ is the set of all subsets of X
- Cartesian product: $X \times Y = \{(a, b) |\ a \in X \wedge b \in Y\}$
- Cardinality: $|X|$ is the size of X
- Subset: $X \subseteq Y$ if $\forall a \in X, a \in Y$
- Set Equality: $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$
- X and Y are **disjoint** if $X \cap Y = \varnothing$

Functions

- A function $f : X \to Y$ is **injective** if $\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$.
- A function $f : X \to Y$ is **surjective** if $\forall y \in Y, \exists x \in X, f(x) = y$.
- A function is **bijective** if it is injective and surjective.

[1] Many problems from *An Infinite Descent into Pure Mathematics* by Clive Newstead

2 Warmup

1. In terms of $|A|$ and $|B|$, what are the following cardinalities? Give exact answers if possible; otherwise give upper and lower bounds.
   (a) $|A \cap B|$
   (b) $|A \cup B|$
   (c) $|A \times B|$
   (d) $|\mathscr{P}(A)|$
   (e) The number of functions from A to B

2. Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9, 10\}$, and $C = \{2, 4, 6, 8\}$. Define a function $f : A \to B$ which is an injection but not a surjection. Define a function $g : B \to A$ which is a surjection but not an injection. Define a function $h : A \to C$ which is neither a surjection nor an injection.

3. For each of the following statements, determine whether it is true for all sets X, Y, false for all sets X, Y, or true for some choices of X and Y and false for others.
   (a) $\mathscr{P}(X \cup Y) = \mathscr{P}(X) \cup \mathscr{P}(Y)$
   (b) $\mathscr{P}(X \cap Y) = \mathscr{P}(X) \cap \mathscr{P}(Y)$
   (c) $\mathscr{P}(X \times Y) = \mathscr{P}(X) \times \mathscr{P}(Y)$
   (d) $\mathscr{P}(X \setminus Y) = \mathscr{P}(X) \setminus \mathscr{P}(Y)$

3 Problems

1. Prove that $X \subseteq Y$ if and only if $X \cap Y = X$.
2. Prove that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.
3. Let X be a set and let $U, V \subseteq X$. Prove that U and V are disjoint if and only if $U \subseteq X \setminus V$.
4. Find a family of sets $\{X_n \mid n \in \mathbb{N}\}$ such that:
   - $\bigcup_{n\in\mathbb{N}} X_n = \mathbb{N}$
   - $\bigcap_{n\in\mathbb{N}} X_n = \varnothing$
   - $X_i \cap X_j \neq \varnothing$ for all $i, j \in \mathbb{N}$
5. Construct a bijection between the following pairs of sets. Prove that your function is a bijection.
   (a) $\mathbb{Z}$ and $\mathbb{N}$
   (b) $\mathbb{N}^+ \times \{0, 1\}$ and $\mathbb{Z} \setminus \{0\}$
   (c) Binary strings of length n and $\mathscr{P}(\{1, 2, ..., n\})$
   (d) $A \times (B \times C)$ and $(B \times A) \times C$ for any sets A, B, C
   (e) $(0, 1)$ and $[0, 1)$
   (f) $(0, 1)$ and $\mathbb{R}$
6. Prove that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
7. Suppose we have a bijection $f : A \to B$. Construct a bijection $g : \mathscr{P}(A) \to \mathscr{P}(B)$.
8. Do the following sets have the same or different cardinalities?
   - $\mathscr{P}(\mathbb{N})$
   - All subsets of $\mathbb{N}$ which have a finite number of elements

4 Further Problems

1. Construct a bijection between the following pairs of sets. Prove that your function is a bijection.
   (a) $(0, 1]$ and $(0, 1]^2$
   (b) $\mathbb{R}$ and $\mathbb{R}^2$
   (c) Infinite binary strings and the Cantor set
2. Prove that $|\mathbb{R}| = |\mathscr{P}(\mathbb{N})|$.
3. Prove that two sets having a bijection between them is an equivalence relation (reflexive, symmetric, transitive).