Review

Varsity Practice 1/17/21
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1 Useful Definitions and Formulas

- A graph G is a *bipartite* graph if you can partition the vertices into two sets X, Y such that all the edges have one endpoint in X and the other in Y.
- A *matching* is a set of edges that pairs up certain vertices in a graph.
- A*perfect matching* is a matching that pairs up all vertices of the graph.
- AM-GM: Given $x_1, ..., x_n$, $\sum x_i/n \leq (\Pi x_i)^{1/n}$.
- Cauchy-Schwarz: Given two vectors u, v, $|u \cdot v|^2 \leq ||u||^2||v||^2$. Alternatively, given $x_1, ..., x_n$ and $y_1, ..., y_n$, you can find that $(\sum x_i y_i)^2 \leq (\sum x_i^2)(\sum y_i^2)$. This can also be expressed as $(\sum \sqrt{x_i y_i})^2 \leq (\sum x_i)(\sum y_i)$, assuming $x_i, y_i \geq 0$.
- Jensen: Let f be a convex function, $t \in (0, 1)$. Then $tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2)$.
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$ $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

2 Problem Set

1. Let G be a graph such that every vertex has the same degree. Must this graph contain a perfect matching? What if we also know the graph: has even number of vertices? is connected? is bipartite?
2. In a class of n students, the teacher asked them to do a project either by yourself, or with another classmate. For some reason, every student only wants to be in a team with their close friends (assume the closeness is mutual). We call a group of students 'popular' if every person outside the group is close with at least one person within the group. Show that the largest number of two person teams the class can form is not larger than the smallest size of a popular group.
3. The bisector of the outer angle at vertex C of $\triangle ABC$ intersects the circumcircle at point D. Prove that $AD = BD$.
4. Prove that $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$ for $a, b, c > 0$.
5. If $2x^2 + 3y^2 + 4z^2 = 1$, find the maximum value of $4x + 3y + 2z$.
6. (2007 AMC 12A #17) Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a - b)$?

7. (1992 USAMO #2)Prove $\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}$. (Hint: multiply both sides by $\sin 1$)

8. (PUMaC 2017) Let the sequence $a_1, a_2, \ldots$ be defined recursively as follows: $a_n = 11a_{n-1} - n$. Find the smallest value of a_0 such that all the terms of the sequence are positive.

9. Show that a tree has at most one perfect matching.

10. The altitude AH is drawn in $\triangle ABC$. Let O be the center of the circumcircle. Prove that $\angle OAH = |\angle B - \angle C|$.

11. Let x, y, z be positive real numbers such that $x + y + z = 1$. Maximize x^3y^2z.

12. (1995 AIME #7) Given that $(1 + \sin t)(1 + \cos t) = 5/4$ and $(1 - \sin t)(1 - \cos t) = \frac{m}{n} - \sqrt{k}$, where $k, m,$ and n are positive integers with m and n relatively prime, find $k + m + n$.

13. Solve the recurrence $b_{n+1} = 3b_n + n$ where $b_0 = 0$.