Transformations of the Plane

JV Practice 11/15/20
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1 Warmup

1.1 Vectors and Converting from Polar to Cartesian Coordinates

1. What is the vector $\vec{v}$ from the point $(1, 6)$ to the point $(-7, 2)$?
2. What is the angle θ (in radians) associated with $\vec{v}$? (This is the angle between the positive x-axis and $\vec{v}$)
3. What is the magnitude m associated with $\vec{v}$?

1.2 Transformations and Matrices

1. What is the reflection of $(3, 5)$ across the line $x = 0$?
2. What is the reflection of $(3, 5)$ across the line $y = -x$?
3. More generally, how would you reflect a vector across the line $x = 0$? Can you represent this in matrix form? What about for the line $y = -x$?
4. What is the length of the projection of $\vec{p} = (3, 5)$ onto the line $y = 2x$?
5. What is the translation of the point $(5, -1)$ by $\vec{v}$?

2 Some Useful Formulas

Cartesian representation of vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Matrix index convention:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

Matrix multiplication on a vector:

$$M\vec{v} = \begin{bmatrix} M_{11}v_1 + M_{12}v_2 \\ M_{21}v_1 + M_{22}v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Dot product:

$$\vec{v} \cdot \vec{w} = v_1 * w_1 + v_2 * w_2 = \text{ scalar}$$

Projection of $\vec{v}$ onto $\vec{w}$:

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||} * \frac{\vec{w}}{||\vec{w}||}$$

Finding the angle between two vectors:

$$||\vec{v}|| * ||\vec{w}|| * \cos(\theta) = \vec{v} \cdot \vec{w}$$

Matrix multiplication:

$$MN = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} M_{11}N_{11} + M_{12}N_{21} & M_{11}N_{12} + M_{12}N_{22} \\ M_{21}N_{11} + M_{22}N_{21} & M_{21}N_{12} + M_{22}N_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

3 Formulas We Will Derive

3.1 Reflection

To reflect a point p across a line with normal vector $\vec{\ell}$, the resulting point q will be:

$$q = 2 * (\vec{p} \cdot \vec{\ell}) * \vec{\ell} - \vec{p}$$

Since $\vec{\ell}$ is a normal vector, we can represent it as $\ell_1 = \cos(\theta)$ and $\ell_2 = \sin(\theta)$. Our formula then becomes:

$$q = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

3.2 Rotation

To rotate a point p around the origin by angle θ, the resulting point q will be:

$$q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

3.3 Translation

To translate a point p by a vector $\vec{v}$, the resulting point q will be:

$$q = \begin{bmatrix} v_1 + p_1 \\ v_2 + p_2 \end{bmatrix}$$

3.4 Glide Reflection

A glide reflection is just a composition of a reflection and translation along the same line.

4 Problems

4.1 Exploring Properties of Transformations

1. Find the rotation matrix for $\theta = \frac{2\pi}{3}$. What is the result when $(2, -6)$ is rotated by $\frac{2\pi}{3}$?

2. Find the reflection matrix for the line $y = 2x$. What is the result when the point $(1, 5)$ is reflected across this line?

3. What is the transformation matrix for rotating by $\frac{3\pi}{4}$ and then reflecting across the line $y = 0$? Does this correspond to any simpler transformation? If so, what?

4. What is the transformation matrix for reflecting across the line at $\frac{2\pi}{3}$ and then across the line at $\frac{\pi}{3}$? Does this correspond to any simpler transformation? If so, what?

5. What is the transformation matrix for reflecting across the line $y = 4$? What if you then reflect across the line $y = 2$? Can you simplify that? What if you then reflect across the line $y = 1$? Can you simplify that?

6. Find the rotation matrix for rotating by $\frac{\pi}{3}$. Now find the rotation matrix for rotating by $-\frac{\pi}{3}$. What do you get when you multiply them?

7. What are the fixed points under a rotation? Under a translation? Under a reflection?

8. Prove that rotations, reflections, translations, and glide reflections preserve collinearity.

4.2 Applying Geometry Concepts to Problems

1. A triangle with vertices $A(0, 2)$, $B(-3, 2)$, and $C(-3, 0)$ is reflected about the x-axis, then the image $\triangle A'B'C'$ is rotated counterclockwise about the origin by $90°$ to produce $\triangle A''B''C''$. What transformation will return $\triangle A''B''C''$ to $\triangle ABC$?

2. A piece of graph paper is folded once so that $(0, 2)$ is matched with $(4, 0)$, and $(7, 3)$ is matched with (m, n). Find (m, n).

3. In the Cartesian plane let $A = (1, 0)$ and $B = \left(2, 2\sqrt{3}\right)$. Equilateral triangle ABC is constructed so that C lies in the first quadrant. Let $P = (x, y)$ be the center of $\triangle ABC$. Then $x \cdot y$ can be written as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is an integer that is not divisible by the square of any prime. Find p,q, and r.

4. The points $(0, 0)$, $(a, 11)$, and $(b, 37)$ are the vertices of an equilateral triangle. Find the value of ab.

5. A bee starts flying from point P_0. She flies 1 inch due east to point P_1. For $j \geq 1$, once the bee reaches point P_j, she turns $30°$ counterclockwise and then flies $j + 1$ inches straight to point P_{j+1}. When the bee reaches P_{2015} she is exactly $a\sqrt{b} + c\sqrt{d}$ inches away from P_0, where a, b, c and d are positive integers and b and d are not divisible by the square of any prime. What is $a + b + c + d$?

6. A parabola with equation $y = ax^2 + bx + c$ is reflected about the x-axis. The parabola and its reflection are translated horizontally five units in opposite directions to become the graphs of $y = f(x)$ and $y = g(x)$, respectively. What is the form of $y = (f + g)(x)$?

7. The parabola with equation $p(x) = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $q(x) = dx^2 + ex + f$. What is $a + b + c + d + e + f$ in terms of h or k or both?

4.3 Other Geometry Problems

1. (2012 AMC 10A Problem 21) Let points $A = (0, 0, 0)$, $B = (1, 0, 0)$, $C = (0, 2, 0)$, and $D = (0, 0, 3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of rectangle $EFGH$?

2. (2018 AMC 12B Problem 23) Ajay is standing at point A near Pontianak, Indonesia, $0°$ latitude and $110°$ E longitude. Billy is standing at point B near Big Baldy Mountain, Idaho, USA, $45°$ N latitude and $115°$ W longitude. Assume that Earth is a perfect sphere with center C. What is the degree measure of $\angle ACB$?

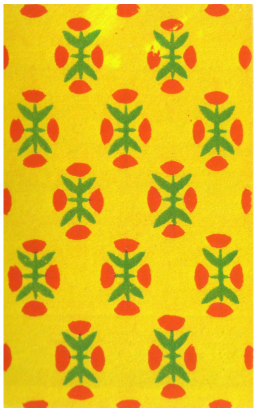
3. (2007 AMC 12B Problem 25) Points A, B, C, D and E are located in 3-dimensional space with $AB = BC = CD = DE = EA = 2$ and $\angle ABC = \angle CDE = \angle DEA = 90^o$. The plane of $\triangle ABC$ is parallel to $\overline{DE}$. What is the area of $\triangle BDE$?

4. (2013 AIME I Problem 7) A rectangular box has width 12 inches, length 16 inches, and height $\frac{m}{n}$ inches, where m and n are relatively prime positive integers. Three faces of the box meet at a corner of the box. The center points of those three faces are the vertices of a triangle with an area of 30 square inches. Find $\frac{m}{n}$.

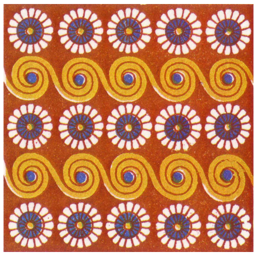
4.4 Wallpaper Patterns

For each of these images, identify all the transformations that leave the image the same. (For example, rotation by 120 degrees, or translation along a specific axis.) Please treat the images as idealized versions of themselves; any small flaws in the artwork should be ignored.

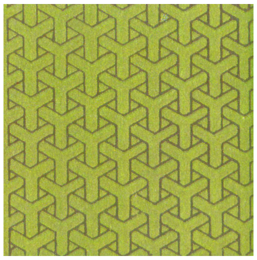
1. 

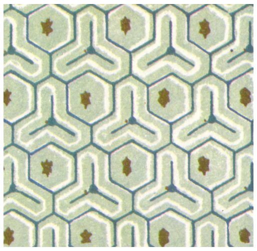
2. 

3. 

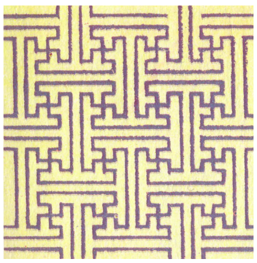
4. 

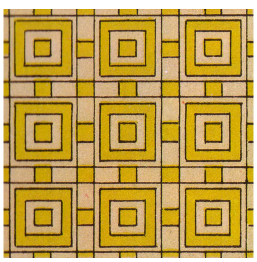
5. 

6. 

7. 

8. 

9. 

10. 

4.5 Fun Resources for Learning More About Transformations

There are other types of transformations of the plane, as well as transformations of higher dimensional spaces. If you want to learn more about general transformations of the plane, look for "Affine Transformations." If you want to go even more general, here is a cool video about Möbious transformations: `https://www.youtube.com/watch?v=0z1fIsUNhO4` If you want to learn more about wallpaper groups, the wikipedia article is a good starting place: `https://en.wikipedia.org/wiki/Wallpaper_group`.