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Geometry inequalities 1

1. Warm-Up

1. Segments AB and CD of length 1 intersect at point O and $\angle AOC = 60°$. Prove that $AC + BD \geq 1$.

2. Cities A, B, C and D are located so that the distance from C to A is less than the distance from D to A, and the distance from C to B is less than the distance from D to B. Prove that the distance from city C to any point on the straight road connecting the cities A and B is less than the distance from city D to this point.

3. All sides of the convex pentagon are equal, and all its angles are different. Prove that the maximum and minimum angles are adjacent to the one side of the pentagon.

4. In an acute-angled triangle, the distance from the middle of any side to the opposite vertex is the sum of the distances from the middle of the same side to the other sides of the triangle. Prove that the triangle is equilateral.

2. Problems

1. Prove that the sum of the diagonals of a convex quadrilateral is smaller than its perimeter, but larger than the half-perimeter.

2. The angle-bisector AD and the height BE are drawn in the acute-angled $\triangle ABC$. Prove that $\angle CED > 45°$.

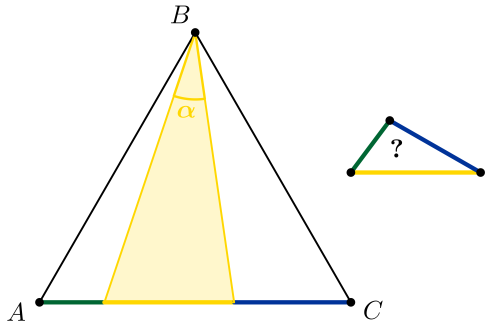
3. Let O be the center of the circumscribed circle of $\triangle ABC$. The points M and N are such points on the sides AB and BC respectively that $2\angle MON = \angle AOC$. Prove that the perimeter of the $\triangle MBN$ is at least the side AC.

4. The point E us such point on the side AD of the convex quadrilateral $ABCD$ that $BE > AB$. We also know that $AC > CD$. Prove that $ED < 2BC$.

5. Let AB be a chord in a circle with center O and OK is such radius that perpendicular to AB and intersect it at point M. Let P be a point on large arc AB that is different from the middle of this arc. Line PM intersects the circle a second time at point Q, and line PK intersects AB at point R. Prove that $KR > MQ$.

6. Two triangles have equal largest sides and equal minimum angles. A new triangle is constructed with sides equal to the sums of the corresponding sides of these triangles: sum of the largest sides of the two triangles, the average sides and the shortest sides. Prove that the area of the new triangle is not less than twice the sum of the areas of the original.

7. A spotlight located at vertex B of an equilateral $\triangle ABC$ illuminates the angle α (look picture). Find all such values $\alpha < 60^\circ$ that for any position of the spotlight when the consecrated corner is entirely inside the $\angle ABC$, you can make a triangle from the consecrated and two unlit sections of the AC side.



3. Bonus

1. Let a, b and c be the lengths of the sides of the triangle, m_a, m_b and m_c be the lengths of the medians drawn to the corresponding sides, and D be the diameter of the circumscribed circle of our triangle. Prove that:
$$\frac{a^2+b^2}{m_c}+\frac{b^2+c^2}{m_a}+\frac{c^2+a^2}{m_b}\leq 6D$$