Miscellaneous Combinatorics 2

V Practice 5/03/20

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1 Adapt. Improvise. Overcome. Warmup.

1. On a 5 by 5 grid we randomly place two cars, which each occupy a single cell and randomly face in one of the four cardinal directions. It is given that the two cars do not start in the same cell. In a move, one chooses a car and shifts it one cell forward. The probability that there exists a sequence of moves such that, afterward, both cars occupy the same cell is m/n where m and n are relatively prime positive integers. Compute $100m + n$.

2. A magician has a hat that contains a white rabbits and b black rabbits. The magician repeatedly draws pairs of rabbits chosen at random from the hat, without replacement. Call a pair of rabbits checkered if it consists of one white rabbit and one black rabbit. Given that the magician eventually draws out all the rabbits without ever drawing out an unpaired rabbit and that the expected value of the number of checkered pairs that the magician draws is 2020, compute the number of possible pairs (a, b).

3. A mahogany bookshelf has four identical-looking books which are 200, 400, 600, and 800 pages long. Velma picks a random book off the shelf, flips to a random page to read, and puts the book back on the shelf. Later, Daphne also picks a random book off the shelf and flips to a random page to read. Given that Velma read page 122 of her book and Daphne read page 304 of her book, the probability that they chose the same book is m/n, for relatively prime positive integers m and n. Compute $100m + n$.

2 OMO? OWO.

All of these problems are taken from various years in the OMO. It's pretty miscellaneous.

1. (OMO W12 19) There are 20 geese numbered 1 through 20 standing in a line. The even numbered geese are standing at the front in the order 2, 4, . . . , 20, where 2 is at the front of the line. Then the odd numbered geese are standing behind them in the order, 1, 3, 5, . . . , 19, where 19 is at the end of the line. The geese want to rearrange themselves in order, so that they are ordered 1, 2, . . . , 20 (1 is at the front), and they do this by successively swapping two adjacent geese. What is the minimum number of swaps required to achieve this formation?

2. (OMO S14 15) In Prime Land, there are seven major cities, labelled $C_0, C_1, ..., C_6$. For convenience, we let $C_{n+7} = C_n$ for each $n = 0, 1, ..., 6$; i.e. we take the indices modulo 7. Al initially starts at city C_0. Each minute for ten minutes, Al flips a fair coin. If the coin land heads, and he is at city C_k, he moves to city C_{2k}; otherwise he moves to city C_{2k+1}. If the probability that Al is back at city C0 after 10 moves is $m/1024$, find m.

3. (OMO F13 11) Four orange lights are located at the points (2, 0), (4, 0), (6, 0) and (8, 0) in the xy-plane. Four yellow lights are located at the points (1, 0), (3, 0), (5, 0), (7, 0). Sparky chooses one or more of the lights to turn on. In how many ways can he do this such that the collection of illuminated lights is symmetric around some line parallel to the y-axis?

4. (OMO S14 9) Eighteen students participate in a team selection test with three problems, each worth up to seven points. All scores are nonnegative integers. After the competition, the results are posted by Evan in a table with 3 columns: the student's name, score, and rank (allowing ties), respectively. Here, a student's rank is one greater than the number of students with strictly higher scores (for example, if seven students score 0, 0, 7, 8, 8, 14, 21 then their ranks would be 6, 6, 5, 3, 3, 2, 1 respectively).

   When Richard comes by to read the results, he accidentally reads the rank column as the score column and vice versa. Coincidentally, the results still made sense! If the scores of the students were $x_1 \leq x_2 \leq ... \leq x_{18}$, determine the number of possible values of the 18-tuple $(x_1, x_2, ..., x_{18})$. In other words, determine the number of possible multisets (sets with repetition) of scores.

5. (OMO F13 16) Al has the cards 1, 2, . . . , 10 in a row in increasing order. He first chooses the cards labeled 1, 2, and 3, and rearranges them among their positions in the row in one of six ways (he can leave the positions unchanged). He then chooses the cards labeled 2, 3, and 4, and rearranges them among their positions in the row in one of six ways. (For example, his first move could have made the sequence 3, 2, 1, 4, 5, . . . , and his second move could have rearranged that to 2, 4, 1, 3, 5, ...) He continues this process until he has rearranged the cards with labels 8, 9, 10. Determine the number of possible orderings of cards he can end up with.

6. (OMO S16 10) Lazy Linus wants to minimize his amount of laundry over the course of a week (seven days), so he decides to wear only three different T-shirts and three different pairs of pants for the week. However, he doesn't want to look dirty or boring, so he decides to wear each piece of clothing for either two or three (possibly nonconsecutive) days total, and he cannot wear the same outfit (which consists of one T-shirt and one pair of pants) on two different (not necessarily consecutive) days. How many ways can he choose the outfits for these seven days?

7. (OMO W13 21) Dirock has a very neat rectangular backyard that can be represented as a 32 bu 32 grid of unit squares. The rows and columns are each numbered 1, 2, . . . , 32. Dirock is very fond of rocks, and places a rock in every grid square whose row and column number are both divisible by 3. Dirock would like to build a rectangular fence with vertices at the centers of grid squares and sides parallel to the sides of the yard such that

   (a) The fence does not pass through any grid squares containing rocks;

   (b) The interior of the fence contains exactly 5 rocks.

   In how many ways can this be done?

8. (OMO W12 30) The Lattice Point Jumping Frog jumps between lattice points in a coordinate plane that are exactly 1 unit apart. The Lattice Point Jumping Frog starts at the origin and makes 8 jumps, ending at the origin. Additionally, it never lands on a point other than the origin more than once. How many possible paths could the frog have taken?

9. (OMO W13 27) Geodude wants to assign one of the integers 1, 2, 3, . . . , 11 to each lattice point (x, y, z) in a 3D Cartesian coordinate system. In how many ways can Geodude do this if for every lattice parallelogram $ABCD$, $(a+c)-(b+d) \equiv 0 \bmod 11$, where a, b, c, d are the values at points A, B, C, D respectively? (A lattice point is a point with all integer coordinates. A lattice parallelogram is a parallelogram with all four vertices lying on lattice points.)

10. (OMO W12 45) Let K_1, K_2, K_3, K_4, K_5 be 5 distinguishable keys, and let D_1, D_2, D_3, D_4, D_5 be 5 distinguishable doors. For $1 \leq i \leq 5$, key K_i opens doors D_i and D_{i+1} (where $D_6 = D_1$) and can only be used once. The keys and doors are placed in some order along a hallway. Key$ha walks into the hallway, picks a key and opens a door with it, such that she never obtains a key before all the doors in front of it are unlocked. In how many such ways can the keys and doors be ordered if Key$ha can open all the doors?

3 The Definition of Insanity

1. Let S be the set of line segments between any two vertices of a regular 21-gon. If we select two distinct line segments from S at random, what is the probability they intersect? Note that line segments are considered to intersect if they share a common vertex.

2. (SMT 2012 AT TB 3) There are 7 cages in a row in an animal shelter and an ample supply of three different kind of animals: dogs, cats, and golden bears. Since golden bears do not like each other, they cannot be placed in adjacent cages. Cats also do not like each other (but not as much so as golden bears), so there cannot be more than two cats in a row. How many ways are there to fill the cages with animals?

3. (CHMMC 2014 Ind 6) Suppose that you start with the number 8 and always have two legal moves:

   - Square the number
   - Add one if the number is divisible by 8 or multiply by 4 otherwise

   How many sequences of 4 moves are there that return to a multiple of 8?

4. (CHMMC 2012 Ind 5) . In an 8 by 8 chessboard, a pawn has been placed on the third column and fourth row, and all the other squares are empty. It is possible to place nine rooks on this board such that no two rooks attack each other. How many ways can this be done? (Recall that a rook can attack any square in its row or column provided all the squares in between are empty.)

5. (SMT 2013 AT 4) Given the digits 1 through 7, one can form $7! = 5040$ numbers by forming different permutations of the 7 digits (for example, 1234567 and 6321475 are two such permutations). If the 5040 numbers obtained are then placed in ascending order, what is the 2013th number?

6. (SMT 2019 Disc 9) Edward has a 3 by 3 tic-tac-toe board and wishes to color the squares using 3 colors. How many ways can he color the board such that there is at least one row whose squares have the same color and at least one column whose squares have the same color? A coloring does not have to contain all three colors and Edward cannot rotate or reflect his board.

7. (CHMMC 2013 Disc 10) A robot starts in the bottom left corner of a 4 by 4 grid of squares. How many ways can it travel to each square exactly once and then return to its start if it is only allowed to move to an adjacent (not diagonal) square at each step?

8. (SMT 2019 Disc TB 1) Find the number of pairs (A, B) of distinct subsets of 1, 2, 3, 4, 5, 6, 7, 8, such that A is a proper subset of B.

9. (BMT 2013 Disc 3) Suppose we have 2013 piles of coins, with the i th pile containing exactly i coins. We wish to remove the coins in a series of steps. In each step, we are allowed to take away coins from as many piles as we wish, but we have to take the same number of coins from each pile. We cannot take away more coins than a pile actually has. What is the minimum number of steps we have to take?

10. (CHMMC 2010 Ind 11) Darryl has a six-sided die with faces 1, 2, 3, 4, 5, 6. He knows the die is weighted so that one face comes up with probability 1/2 and the other five faces have equal probability of coming up. He unfortunately does not know which side is weighted, but he knows each face is equally likely to be the weighted one. He rolls the die 5 times and gets a 1, 2, 3, 4 and 5 in some unspecified order. Compute the probability that his next roll is a 6.

11. (CHMMC 2014 Ind 12) A 5x5 grid is missing one of its main diagonals. In how many ways can we place 5 pieces on the grid such that no two pieces share a row or column?

12. A Nishop is a chess piece that moves like a knight on its first turn, like a bishop on its second turn, and in general like a knight on odd-numbered turns and like a bishop on even-numbered turns. A Nishop starts in the bottom-left square of a 3 by 3-chessboard. How many ways can it travel to touch each square of the chessboard exactly once?

4 I'm Epic Gamer

1. Two kids A and B play a game as follows: From a box containing n marbles ($n > 1$), they alternately take some marbles for themselves, such that:

   1. A goes first.

   2. The number of marbles taken by A in his first turn, denoted by k, must be between 1 and n, inclusive.

   3. The number of marbles taken in a turn by any player must be between 1 and k, inclusive.

   The winner is the one who takes the last marble. What is the sum of all n for which B has a winning strategy?

2. Two people, A and B, are playing a game with three piles of matches. In this game, a move consists of a player taking a positive number of matches from one of the three piles such that the number remaining in the pile is equal to the nonnegative difference of the numbers of matches in the other two piles. A and B each take turns making moves, with A making the first move. The last player able to make a move wins. Suppose that the three piles have 10, x, and 30 matches. Find the largest value of x for which A does not have a winning strategy.

3. Two bored millionaires, Bilion and Trilion, decide to play a game. They each have a sufficient supply of \$1, \$2, \$5, and \$10 bills. Starting with Bilion, they take turns putting one of the bills they have into a pile. The game ends when the bills in the pile total exactly \$1,000,000, and whoever makes the last move wins the \$1,000,000 in the pile (if the pile is worth more than \$1,000,000 after a move, then the person who made the last move loses instead, and the other person wins the amount of cash in the pile). Assuming optimal play, how many dollars will the winning player gain?

4. Amandine and Brennon play a turn-based game, with Amadine starting. On their turn, a player must select a positive integer which cannot be represented as a sum of nonnegative multiples of any of the previously selected numbers. For example, if 3, 5 have been selected so far, only 1, 2, 4, 7 are available to be picked; if only 3 has been selected so far, all numbers not divisible by three are eligible. A player loses immediately if they select the integer 1. Call

a number n feminist if gcd(n, 6) $= 1$ and if Amandine wins if she starts with n. Compute the sum of the feminist numbers less than 40.

5. You are playing a game in which you have 3 envelopes, each containing a uniformly random amount of money between 0 and 1000 dollars. (That is, for any real $0 \leq a < b \leq 1000$, the probability that the amount of money in a given envelope is between a and b is $b - a/1000$.) At any step, you take an envelope and look at its contents.

   You may choose either to keep the envelope, at which point you finish, or discard it and repeat the process with one less envelope. If you play to optimize your expected winnings, your expected winnings will be E. What is $\lfloor E \rfloor$, the greatest integer less than or equal to E?

6. Near the end of a game of Fish, Celia is playing against a team consisting of Alice and Betsy. Each of the three players holds two cards in their hand, and together they have the Nine, Ten, Jack, Queen, King, and Ace of Spades (this set of cards is known by all three players). Besides the two cards she already has, each of them has no information regarding the other two's hands (In particular, teammates Alice and Betsy do not know each other's cards). It is currently Celia's turn. On a player's turn, the player must ask a player on the other team whether she has a certain card that is in the set of six cards but not in the asker's hand. If the player being asked does indeed have the card, then she must reveal the card and put it in the asker's hand, and the asker shall ask again (but may ask a different player on the other team); otherwise, she refuses and it is now her turn. Moreover, a card may not be asked if it is known (to the asker) to be not in the asked person's hand.

   The game ends when all six cards belong to one team, and the team with all the cards wins. Under optimal play, the probability that Celia wins the game is p/q, for relatively prime positive integers p and q. Find $100p + q$.