Pigeonhole Principle II

JV Practice 5/17/20

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1 Warmup

1. 31 balls, some black and some white, are divided into five jars. What is the largest guaranteed number of balls of the same colour in the same jar?
2. The integers $1, 2, \ldots, 10$ are written on a circle, in any order. Show that there are 3 adjacent numbers whose sum is 17 or greater.
3. Among a group of 6 people, where every pair of people are either friends or enemies, prove that there are either three people who are all friends, or 3 people who are all enemies. If you are having trouble, try drawing some pictures.

2 Warmup if You Missed Last Week

Use the Pigeonhole Principle: if you have k items (pigeons) into m boxes (pigeonholes), there must be at least $\lceil \frac{k}{m} \rceil$ items in one of the boxes.[1]

1. If I give 100 candy bars to 15 students, at least how many candy bars will the happiest student have? (Every student measures happiness directly in terms of number of candy bars received.)
2. How many students do I need to poll in order to guarantee two students have the same birthday?
3. If I have 7 points in a hexagon with side length 1, what is the smallest distance x such that I can guarantee that some two points are at most distance x apart?

3 Proof Practice

Last practice, I had you submit some proofs via the usual Google form. This week, let's focus on one problem specifically to develop your proof writing skills. (We will have a different Google form to submit this to, in addition to the usual form.) Try to make your proof as rigorous and detailed as possible, so that there are no steps where you make an assertion that is not backed up by a reason.

1. Let A be any set of exactly 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 97, 100$. Prove there must be at least two distinct integers in A whose sum is 104.

[1] If you aren't familiar with them, the funny brackets mean to take the ceiling, aka round up to the nearest integer.

4 Problems

1. An equilateral triangle with side length 1 has 10 points in it. What is the smallest distance x such that you are guaranteed to have two points of distance less than or equal to distance x?

2. If 9 people are seated in a row of 12 chairs, then how many consecutive chairs can you guarantee will be filled with people?

3. If there is a party, show that two people at the party know the same number of people. (Parties must have at least two people.)

4. Suppose your younger sibling, who can't read, dropped your clock on the floor and all the numbers fell off. They glued them all back on randomly and hung it back on the wall in an effort to be helpful, but got all the numbers wrong. What is the maximum number of numbers you will be able to guarantee are in the right place for some rotation of the clock?

5. Ten students took a test, and got a total sum of 35 points. Some students got 1, 2, and 3 points. What is the lowest possible high score among the students?

6. CMU's Scotty dog colors every point on the plane either red or black. Prove that no matter how the coloring is done, there must exist two points, exactly a mile apart, that are the same color.

7. Given any set of 100 integers, what is the largest number of integers that can be chosen such that all the pairwise differences of the chosen numbers are guaranteed to be divisible by 7?

8. Let $1^{(k)}$ represent that number that is k digits long, all of which are ones. For example, $1^{(5)} = 11111$. Prove that there is a number in the set $S = \{1^{(1)}, 1^{(2)}, 1^{(3)}, \dots\ 1^{(222)}\}$ that is divisible by 221.

9. How many points in the plane with integer coordinates do you need to guarantee that the midpoint of some two of the points has integer coordinates?

10. How many points placed randomly on a sphere do you need to guarantee that there exists a half sphere containing 6 of the points? (Points on the border count as in both halves.) Hint in footnote.[2]

11. Given n integers, prove that some nonempty subset of them has sum divisible by n.

[2] Hint: 2 points on a sphere define a great circle, which cuts a sphere into two halves.