Sequences and Series I

JV Practice 4/26/20

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Definitions

An **arithmetic sequence** $a_1, a_2, \ldots, a_n$ is a sequence for which there is a fixed constant d such that $a_{i+1} = a_i + d$ for all i. The constant d is called the *common difference* of the sequence. For example, $-5, -1, 3, 7, 11, 15, 19$ is an arithmetic sequence with common difference 4.

A **geometic sequence** $a_1, a_2, \ldots, a_n$ is a sequence for which there is a fixed constant r such that $a_{i+1} = r \cdot a_i$ for all i. The constant d is called the *common ratio* of the sequence. For example, $96, 48, 24, 12, 6, 3, \frac{3}{2}$ is a geometric sequence with common ratio $\frac{1}{2}$.

Warm-up Problems

1. Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?
2. The geometric series $a + ar + ar^2 \ldots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $a + r$?

Week 1 Problems

1. How many terms are there in the arithmetic sequence 43, 46, 49, . . ., 370, 373?
2. The first four terms of an arithmetic sequence are p, 9, $3p-q$, and $3p+q$. What is the 2020^{th} term of this sequence?
3. Write the following repeating decimals as fractions in reduced terms
   (a) $0.\overline{567}$
   (b) $2.1\overline{35}$
4. The terms of an arithmetic sequence add to 715. The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle term of the original sequence.
5. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?
6. Let $a + ar_1 + ar_1^2 + ar_1^3 + \cdots$ and $a + ar_2 + ar_2^2 + ar_2^3 + \cdots$ be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is r_1, and the sum of the second series is r_2. What is $r_1 + r_2$?

7. The real numbers c, b, a form an arithmetic sequence with $a \geq b \geq c \geq 0$. The quadratic $ax^2 + bx + c$ has exactly one root. What is this root?

8. Compute the sum
$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} + \ldots$$

9. A high school basketball game between the Raiders and Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

10. Arithmetic sequences (a_n) and (b_n) have integer terms with
$$a_1 = b_1 = 1 < a_2 \leq b_2$$
and $a_n b_n = 2010$ for some n. What is the largest possible value of n?

11. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $a_1 \cdot a_2 \cdot a_3 \cdots a_{12} = 8^{2020}$, find the number of possible ordered pairs (a, r).

12. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $1/8$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.