Modular Arithmetic

JV Practice 11/17/19

Da Qi Chen

1 Warm-Up

1. Prove that every year (including leap years) has at least one Friday the 13th. What is the most number of Friday the 13th a year can have?
2. What is the last two digits of 11^{38}?
3. Find all possible remainders of $2^{n+2} + 3^{2n+1}$ when divided by 7 where n is a natural number.
4. (AIME 10, 02) The two-digit numbers from 19 to 92 are written consecutively to create the number 19202122...909192. Find the largest integer n such that 3^n divides the above number.

2 Problem Set

1. Prove if $9|(a^3 + b^3 + c^3)$ then $3|abc$.
2. Prove that the sum of the digits of a perfect square can never be 2019.
3. Prove that $7|4^{(2^n)} + 2^{(2^n)} + 1$ for all natural number n.
4. Prove that for any integer $n > 1$, $n^n - n^2 + n - 1$ is divisible by $(n-1)^2$.
5. Prove that $2^k|N$ if and only if the last k digits of N is divisible by 2^k.
6. Suppose $n!! = n!(\frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!})$. Prove that $n!! \equiv n!(\mod n-1)$ for integer $n > 3$.
7. Find the remainder of $6^{83} + 8^{83}$ divided y 49.

1. Find the remainder of 6^{2019} when divided by 37.
2. Find the unit digit of 7^{7^7}.

1. Find the perfect squares in modulo 9.
2. Find all, if any, pairs (x, y) such that $x^2 + 9y^2 = 8$.

1. (AIME 1994) The sequence $3, 15, 24, 48, ...$ are all multiples of 3 that are also one less than a perfect square. Find the last three digits of the 1994th term in the sequence.
2. How many positive integers n strictly less than 40 such that $n^2 + 15n + 122$ is divisible by 6.
3. What is the last non-zero digit of 20!?

1. (IMO 75) Let A be the sum of digits of 4444^{4444} written in decimal form. Let B be the sum of digits of A in decimal form. What is the sum of digits of B?

1. What is 12^{2019} modulo 78?
2. Let S be the set of all positive integers less than 1000 such that when written in binary uses at most two 1's. If a number is chosen from S uniformly at random, what is the probability that it is divisible by 9?