Functional Equations

Varsity Practice 04/12/2020

Anish Sevekari

1 Warm-Up Problems

1. Consider the recurrence $a_{n+1} = 2a_n + 1$, with $a_0 = 0$.
   (a) Compute a_1, a_2, a_3, a_4, a_5. Can you guess the value of a_n?
   (b) Prove that $a_n = 2^n - 1$.
2. Consider the recurrence $a_{n+1} = 2a_n + n$, with $a_0 = 1$. Compute $a_1, a_2, \ldots, a_5$. Can you guess the formula for a_n?
3. Let A_n denote the number of ways of tiling a $1 \times n$ box using blocks of size 1×1 and 1×2.
   (a) Find A_1, A_2, A_3, A_4, A_5. Is the sequence familier to you?
   (b) Compute a recursive expression for A_n. That is, express A_n in terms of A_{n-1} and A_{n-2}.
   (c) (Bonus:) Can you find a closed form formula for A_n?

2 Introduction on Generating functions

A generating function for the series $a_0, a_1, \ldots$ is a power series given by

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{n} a_i x^i$$

Sometimes, recurrence relations between a_i can translate into algebraic identities for $A(x)$, allowing us to compute a closed form expression for $A(x) = \frac{P(x)}{Q(x)}$ for some polynomial $P(X), Q(x)$. Then using partial fraction decomposition for Q, we can infact get closed forms for $A(x)$, in certain cases.

Some useful identities:

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x_i = 1 + x + x^2 + \cdots$$

$$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i$$

$$\frac{dA(x)}{dx} = \sum_{i=0}^{\infty} a_{i+1}(i+1)x^i$$

Note that in the second formula, $\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}$. Note that if n is a natural number, then $\binom{n}{i} = 0$ for $i > n$, which implies that binomal theorem terminates in finitely many terms. The third formula tells you that you can differentiate power series term by term. Note that the equalities hold on a formal level. Loosely speaking, that implies that both sides are equal if they are well defined.

3 Problems (day 1)

1. Find the generating functinos in simple form for the sequences
   (a) $a_n = 1$
   (b) $a_n = n$
   (c) $a_n = \alpha n + \beta$.
2. Use generating functions to solve the recurrent $a_{n+1} = 2a_n + 1$.
3. Consider the Fibonacci Sequence given by the recurrence $A_n = A_{n-1} + A_{n-2}$, with $A_0 = 1, A_1 = 1$.
   (a) Use the recurrence above to get an identity for
   $$F(x) = \sum_{i=1}^{\infty} A_n x^n$$
   (b) Solve the indentity to get a simple form expression for $F(x)$.
   (c) Use partial fraction decomposition to find a closed form value for A_n.

4 Problems (day 2)

1. Catalan Numbers: Let C_n denote the number of pahts from $(0,0)$ to (n,n) consisting of exactly $2n$ steps, each of length 1, either along positive X direction or along positive Y directions, which **do not** go above the diagonal given by $x = y$. (The paths can touch the diagonal).
   (a) Use a combinatorial arguement to show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.
   (b) Show that C_n is also the number of words of length $2n$, made from two symbols, a, b such that no initial string has more b's than a's.
   (c) Show that C_n is also the number of adding parethesis in a string of length $n+1$ which is valid.
   (d) Find a recurrence relatoin for C_n.
   (e) Convert the recurrence relation to find closed form for $A(x)$, the generating function of C_n.
   (f) Use the simple form of $A(x)$ to get a closed form for C_n.
2. Let $\overline{p}_k(n)$ denote the number of partitions of n into at most k parts, where partition of n of size k is a sequence $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 + \ldots + \lambda_k = n$ and $_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. Find a generating function for $\overline{p}_k(n)$. Can you use this to find a generating function for $p(n)$, the total number of partitions of n?