Expected Value

V Practice 3/1/20

Matthew Shi

1 Preliminaries

A *random variable* is a variable whose numerical value is determined by some random process. For example, a die roll or the number of heads seen when flipping 100 coins.

The *expected value* of a random variable is a fancy word for its average value. More formally, if a random variable X has values from a set S, then the expected value of X is

$$\mathbb{E}[X] := \sum_{s \in S} s \cdot \mathbb{P}[X = s].$$

For example, the expected value of a die roll is

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5.$$

The numbers $1, 2, 3, 4, 5, 6$ are the possible outcomes of a die roll, and each occurs with probability $\frac{1}{6}$.

There are two very important facts about expected value.

1. If X and Y are two random variables, then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. For example, the expected sum of two die rolls is $3.5 + 3.5 = 7$.
2. If X and Y are two *independent* random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. For example, the expected product of two separate dice rolls is $3.5 \cdot 3.5 = 12.25$. However, this is only true for independent random variables.

2 Warm-Up Problems

1. Suppose I flip a fair coin n times. Let X, a random variable, represent the number of heads that I flip. Calculate $E[X]$ and the variance of X: $E[X^2] - E[X]^2$.
2. Now suppose I flip a weighted coin n times, with probability $p \in [0, 1]$ of flipping heads. Calculate $E[X]$ and the variance of X: $E[X^2] - E[X]^2$.
3. Suppose there are n people sitting in a circle; each round, every person either turns to their left or right. If two people face each other, they play rock paper scissors. The loser then exits the circle.
   (a) Find the expected number of people that leave each round.
   (b) Find a formula for the expected number of rounds until there's only 1 person left in the circle.

3 Problems

1. (NIMO Fall 2019) An ant starts at the origin of the Cartesian coordinate plane. Each minute it moves randomly one unit in one of the directions up, down, left, or right, with all four directions being equally likely; its direction each minute is independent of its direction in any previous minutes. It stops when it reaches a point (x, y) such that $|x| + |y| = 3$. The expected number of moves it makes before stopping can be expressed as m/n for relatively prime positive integers m and n. Compute $100m + n$.

2. (a) Suppose I roll two dice simultaneously. Calculate the expected number of rolls until I first roll two sixes.
   (b) Suppose I roll one die repeatedly. Calculate the expected number of rolls until I've rolled two sixes in a row.

3. (HMMT Combo 2009) Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

4. (AIME I 2016) Freddy the frog is jumping around the coordinate plane searching for a river, which lies on the horizontal line $y = 24$. A fence is located at the horizontal line $y = 0$. On each jump Freddy randomly chooses a direction parallel to one of the coordinate axes and moves one unit in that direction. When he is at a point where $y = 0$, with equal likelihoods he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

5. (OMO Spring 2018) A mouse has a wheel of cheese which is cut into 2018 slices. The mouse also has a 2019-sided die, with faces labeled 0, 1, 2, . . . , 2018, and with each face equally likely to come up. Every second, the mouse rolls the dice. If the dice lands on k, and the mouse has at least k slices of cheese remaining, then the mouse eats k slices of cheese; otherwise, the mouse does nothing. What is the expected number of seconds until all the cheese is gone?

6. (OMO Fall 16) For her zeroth project at Magic School, Emilia needs to grow six perfectly-shaped apple trees. First she plants six tree saplings at the end of Day 0. On each day afterwards, Emilia attempts to use her magic to turn each sapling into a perfectly-shaped apple tree, and for each sapling she succeeds in turning it into a perfectly-shaped apple tree that day with a probability of 1/2. (Once a sapling is turned into a perfectly-shaped apple tree, it will stay a perfectly-shaped apple tree.) The expected number of days it will take Emilia to obtain six perfectly-shaped apple trees is m/n, for relatively prime positive integers m and n. Find $100m + n$.

7. (OMO Fall 14) Two ducks, Wat and Q, are taking a math test with 1022 other ducklings. The test has 30 questions, and the nth question is worth n points. The ducks work independently on the test. Wat gets the nth problem correct with probability $1/n^2$ while Q gets the nth problem correct with probability $1/n+1$. Unfortunately, the remaining ducklings each answer all 30 questions incorrectly. Just before turning in their test, the ducks and ducklings decide to share answers! On any question which Wat and Q have the same answer, the ducklings

change their answers to agree with them. After this process, what is the expected value of the sum of all 1024 scores?

8. (OMO Spring 15) Alex starts with a rooted tree with one vertex (the root). For a vertex v, let the size of the subtree of v be $S(v)$. Alex plays a game that lasts nine turns. At each turn, he randomly selects a vertex in the tree, and adds a child vertex to that vertex. After nine turns, he has ten total vertices. Alex selects one of these vertices at random (call the vertex v_1). The expected value of $S(v_1)$ is of the form m/n, for relatively prime positive integers m, n. Find $m + n$.

   Note: In a rooted tree, the subtree of v consists of its indirect or direct descendants (including v itself).

9. (OMO Winter 12) You are playing a game in which you have 3 envelopes, each containing a uniformly random amount of money between 0 and 1000 dollars. (That is, for any real $0 \leq a < b \leq 1000$, the probability that the amount of money in a given envelope is between a and b is $(b - a)/1000$. At any step, you take an envelope and look at its contents. You may choose either to keep the envelope, at which point you finish, or discard it and repeat the process with one less envelope. If you play to optimize your expected winnings, your expected winnings will be E. What is $\lfloor E \rfloor$, the greatest integer less than or equal to E?

10. (OMO Fall 18) Ann and Drew have purchased a mysterious slot machine; each time it is spun, it chooses a random positive integer such that k is chosen with probability 2^{-k} for every positive integer k, and then it outputs k tokens. Let N be a fixed integer. Ann and Drew alternate turns spinning the machine, with Ann going first. Ann wins if she receives at least N total tokens from the slot machine before Drew receives at least $M = 2^{2018}$ total tokens, and Drew wins if he receives M tokens before Ann receives N tokens. If each person has the same probability of winning, compute the remainder when N is divided by 2018.

11. (HMMT Combo 20) Anne-Marie has a deck of 16 cards, each with a distinct positive factor of 2002 written on it. She shuffles the deck and begins to draw cards from the deck without replacement. She stops when there exists a nonempty subset of the cards in her hand whose numbers multiply to a perfect square. What is the expected number of cards in her hand when she stops?

12. (HMMT Guts 10) Pick a random integer between 0 and 4095, inclusive. Write it in base 2 (without any leading zeroes). What is the expected number of consecutive digits that are not the same (that is, the expected number of occurrences of either 01 or 10 in the base 2 representation)?

13. (HMMT Guts 10) In the Democratic Republic of Irun, 5 people are voting in an election among 5 candidates. If each person votes for a single candidate at random, what is the expected number of candidates that will be voted for?

14. (OMO Spring 15) Suppose we have 10 balls and 10 colors. For each ball, we (independently) color it one of the 10 colors, then group the balls together by color at the end. If S is the expected value of the square of the number of distinct colors used on the balls, find the sum of the digits of S written as a decimal.