Advanced Counting

V Practice 2/23/20

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1 Warm-Up Problems

1. How many positive integer solutions are there to the system $\Sigma_{i=1}^{k} x_i = n$? What if we count non-negative solutions instead?

2. Prove that $\sum_{a+b=k} \binom{m}{a} + \binom{n}{b} = \binom{m+n}{k}$.

3. (USAMO 1978) Among a group of nine people, every group of three contains at least two people who speak the same language. If every person speaks at most three languages, prove that there exists a group of three people that know a common language.

4. If I drank at least one cup of coffee everyday and at most 36 cups in the past 3 weeks, prove that there exists a consecutive period where I drank exactly 21 cups.

2 Problem Set

1. Show that there exists a positive integer n such that 111...111 (n 1's) is divisible by 2019.

2. (USAMO 1979) Let $A_1, ..., A_{n+1}$ be 3-element subsets of $[n]$. Prove tat there exists $i, , j$ such that $|A_i \cap A_j| = 1$.

3. Can a 4×11 chessboard be covered by L-shaped tetrominoes?

4. Let m, n be positive integers. Let $p(m, n)$ be the number of partitions n into m parts and $q(m, n)$ be the number of partitions of n whose largest part is m. Prove that $p(m, n) = q(m, n)$.

5. Let T_n be the number of non-empty subsets of $[n]$ whose average is an integer. Prove that $T_n - n$ is always even.

6. Prove that for any set of n integers, it contains a subset whose sum is divisible by n.

7. (MOP 2006) Let $n > 1$ be an integer. Let S be a set of real numbers with less than n elements. Suppose that $2^0, 2^1, ..., 2^{n-1}$ can be written as sum of distinct elements of S. Prove that S contains at least one negative number.

8. (IMO 11972) Prove that from a set of ten distinct two digit numbers, there exists two disjoint subsets whose members have the same sum.

9. (MOP 2006) Suppose there are n bags of balls and the weight of any ball is a perfect power of 2. If every bag has the exact same weight, we are guaranteed that there exist at least k_n balls of a particular weight. What is the maximum value of k_n?

10. Prove in a set of seven real numbers, there exists a, b such that $0 \leq \frac{a-b}{1+ab} \leq \frac{1}{\sqrt{3}}$

11. Let $A \subset [200]$ where $|A| \geq 100$. Prove that there exist $a, b \in A$ such that $a|b$.

12. (USAMO 98) A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

13. (IMO 2006) Let M be an m element subset of $[n]$ where $m \geq 3$. Prove that there exists a subset $A \subseteq M$ and $B = \{b, c\} \subset [n]$ where $|A| \geq \frac{m(m-1)(m-2)}{3(n-1)(n-2)}$ and for every $a \in A$, $\{a, a+b, a+c\} \subseteq M$.