# Circles

## JV Practice (Tangents)

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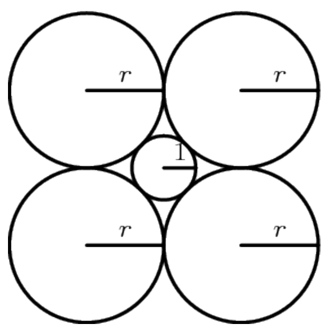
## 1 JV Warm Up

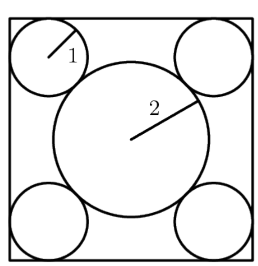
1. (ARML 1983) The length of a common internal tangent to two circles is 7, and a common external tangent is 11. Compute the product of the radii of the two circles.
2. (AMC 10 2004) Square ABCD has side length 2. A semicircle with diameter $AB$ is constructed inside the circle, and the tangent from $C$ intersects side $AD$ at $E$. What is the length of $CE$?
3. (AMC 10 2004) Cirlces $A, B, C$ are externally tangent to each other and internally tangent to circle $D$. Circles $B$ and $C$ are congruent. Circle $A$ has radius 1 and passes through the center of $D$. Compute the radius of cirlce $B$.

## 2 Problems

1. (AMC 10 2007) A circle of radius 1 is surrounded by 4 circles of radius $r$. What is $r$?




2. (AMC 10 2006) Circles with centers $O$ and $P$ have radii 2 and 4, respectively, and are externally tangent. Points $A$ and $B$ on the circle with center $O$ and points $C$ and $D$ on the circle with center $P$ are such that $AD$ and $BC$ are common external tangents to the circles. What is the area of the concave hexagon $AOBCPD$?
3. (AMC 10 2007) Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. What is the area of the square?



4. (AMC 10 2007) Circles centered at $A$ and $B$ each have radius $r$. Points $E$ and $F$ are on the two circles such that $EF$ is a common external tangent. Prove that $ABFE$ is a rectangle.

5. (ARML 1982) Two lines intersect and form an acute angle at a point $A$. Three circles are tangent to these two lines, and successively tangent to each other as shown (see the board). The distance of the center of the largest circle from $A$ is 9 times the distance of the center of the smallest circle from $A$. What is the sine of the angle $A$?

6. (AMC 10 2008) Points $A$ and $B$ lie on a circle centered at $O$, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

7. (AMC 10 2004) Three circles of radius 1 are externally tangent to each other and internally tangent to circle $\omega$. Compute the radius of $\omega$.

8. (AMC 10 2010) Two circles lie outside regular hexagon $ABCDEF$. The first is tangent to $\overline{AB}$, and the second is tangent to $\overline{DE}$. Both are tangent to lines $BC$ and $FA$. What is the ratio of the area of the second circle to that of the first circle?

9. (ARML 1987) From point P outside circle $O$, tangent segment $PA$ and secant $PBC$ are drawn. If $PA = \sqrt{f} + \sqrt{g}$, $OA = 1$ and $m\angle APC = 75°$, then secant $PBC$ will equal 2. Compute the ordered pair of positive integers $(f, g)$, where $f < g$.

10. (AMC 10 2004) Three mutually tangent spheres of radius 1 lie on a horizontal plane. A sphere of radius 2 rests on them. Compute the distance from the plane to the top of the larger sphere.

11. (UMD II 2018) Two tangents $AB$ and $AC$ are drawn to a circle from an exterior point $A$. Let $D$ and $E$ be the midpoints of the line segments $AB$ and $AC$. Prove that the line $DE$ does not intersect the circle.

## 3 Challenge Problems

1. (MathPrize 2017) Circle $\omega_1$ with radius 3 is inscribed in a strip $S$ having border lines $a$ and $b$. Circle $\omega_2$ within $S$ with radius 2 is tangent externally to circle $\omega_1$ and is also tangent to line $a$. Circle $\omega_3$ within $S$ is tangent externally to both circles $\omega_1$ and $\omega_2$, and is also tangent to line $b$. Compute the radius of circle $\omega_3$.

2. (UMD II 2018) A strip is defined to be the region of the plane lying on or between two parallel lines. The width of the strip is the distance between the two lines. Consider a finite number of strips whose widths sum to a number $d < 1$, and let $D$ be a circular closed disk of diameter 1. Prove or disprove: no matter how the strips are placed in the plane, they cannot entirely cover the disk $D$.[1]

[1] Not really related to practice, but a fun problem!

# Varsity Practice (Circles and Cyclic Quadrilaterals)

Anish Sevekari

## 4 Warm up

1. (AMC 10 2009) Points $A$ and $C$ lie on the circle with center $O$. Let $B$ be point such that $BA$ and $BC$ are tangent to the circle. Let $D$ be the point where segment $BO$ intersects the circle. If $\triangle ABC$ is equilateral, find the ratio $\frac{BD}{BO}$.

2. Let $\square ABCD$ be a cyclic quadrilateral with points $A, B, C, D$ on a circle with center $O$ in anticlockwise order. Suppose $AC$ and $BD$ intersect inside the circle at point $E$. Given that $\angle AOB = x$ and $\angle COD = y$, compute $\angle AEB$. Assume that the anticlockwise arcs $AB$ and $CD$ are minor arcs.

3. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be three circles. Show that the pairwise common chords of these three circles are concurrent. (If two circles intersect at points $A$ and $B$, then $\overline{AB}$ is the common chord of the two circles.)

## 5 Problems

1. Four disks with disjoint interiors are mutually tangent. Three of them are equal in size and fourth on is smaller. Find the ratio of radius of smaller disk to one of the larger disks.

2. Let $H$ be the orthocenter of acute angled triangle $\triangle ABC$. Suppose $\overline{AH}$ intersects $\overline{BC}$ at $D$ and circumcircle of $\triangle ABC$ again at $P$. Let $A'$ be the midpoint of $\overline{BC}$. Let $\overrightarrow{HA'}$ intersect the circumcircle at $Q$. Prove that $HA' = A'Q$ and $HD = DP$

3. Let $P$ be a point on circumcircle of triangle $\triangle ABC$. Let $D, E, F$ be the feet of perpendicular from $P$ on $BC, AC, AB$ respectively. Prove that $D, E, F$ are collinear.

4. (SMT 2009) Right triangle $\triangle ABC$ is inscribed in a circle $W$. $\angle A = 65^\circ$ and $\angle B = 25^\circ$. The median from $C$ to $AB$ intersects $W$ again at $D$. Tangents to $W$ at $A$ and $D$ intersect at $P$. Compute $\angle APD$

5. (SMT 2011) Let $ABCD$ be four points on a circle, such that $AB = 6$, $BC = 12$, $CD = 3$, $AD = 6$. Let $E$ be intersection of $AB$ and $CD$ and let $F$ be intersection of $AD$ and $BC$. Compute $EF$.

6. (SMT 2010) A,B,C,D are points along a circle, in that order. $AC$ intersects $BD$ at $X$. If $BC = 6$, $BX = 4$, $XD = 5$ and $AC = 11$, find $AB$.

## 6 Challenge Problems

1. Let $P$ be a point in the interior of triangle $ABC$ (with $CA \neq CB$). The lines $AP$, $BP$, $CP$ intersect the circumcircle of $\triangle ABC$ again at points $K, L, M$ resp. The line tangent to the circumcircle at $C$ intersects $AB$ at $S$. If $SC = SP$ then show that $MK = ML$

2. Let $ABCD$ be a parallelogram with $m\angle A > 90$. Point $E$ lies on $\overrightarrow{DA}$ such that $\overline{BE} \perp \overline{AD}$. The circumcircle of $\triangle ABC$ and $\triangle CDE$ intersect at points $F$ and $C$. Given that $AD = 35$, $DC = 48$ and $CF = 50$, compute $AC$.