Angle Chasing and Similarity

JV Practice

C.J. Argue

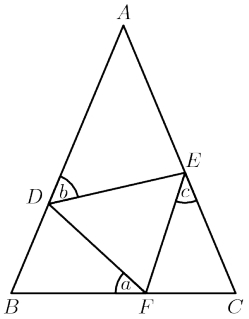
1 Techniques

1. Complementary, supplementary angles.
2. Vertical angles.
3. Angles from parallel lines.
4. Sum of angles in a triangle, general polygon.
5. Isoceles triangle.

2 Warm-Up Problems

1. Show that the sum of the angles of a triangle is 180°.
2. (David Altizio) Suppose $ABCD$ is a convex quadrilateral with $\angle ABC = 144^\circ$, $\angle ADC = 105^\circ$, and $AB = BD = DC$. Compute $\angle BCD - \angle BAD$.
3. (AHSME 1960) In the diagram below AB and AC are the equal sides of an isosceles $\triangle ABC$, in which is inscribed equilateral $\triangle DEF$. Designate $\angle BFD$ by a, $\angle ADE$ by b, and $\angle FEC$ by c. Which of the following is true?

   (A) $b = \frac{a+c}{2}$ **(B)** $b = \frac{a-c}{2}$ **(C)** $a = \frac{b-c}{2}$ **(D)** $a = \frac{b+c}{2}$ **(E)** none of these



3 Problems

1. Find a formula for the sum of the angles of a polygon with n sides. (You may know the formula, but can you prove it?) Does this formula hold for concave polygons?

2. In $\triangle ABC$ there is a point X on side BC such that $AX = BX = CX$. Compute $\angle BAC$.

3. Show that an external angle of a triangle equals the sum of the two other angles in the triangle. This is a handy shortcut, use it in the problems that follow!

4. (UMD 2009) In triangle ABC, let D be the intersection point of the bisector of angle B and the bisector of angle C. If $\angle A = 70°$, what is $\angle CDB$?

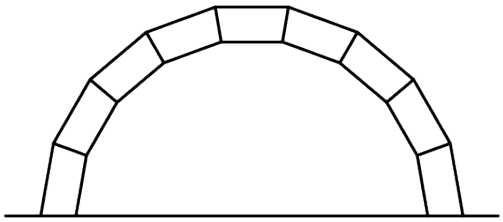
5. (AMC 10 2007) The angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number?

6. (ARML 2008) [See board for figure]. Line k is parallel to line j and $\angle 1, \angle 2, \angle 3$ and $\angle 4$ form, in that order, an increasing arithmetic sequence of angles with $\angle 3 < 90°$. If $\angle 1 = 50°$, compute the largest possible integer value for the measure of $\angle 4$.

7. (MathLeague HS) In a certain quadrilateral, the three shortest sides are congruent, and both diagonals are as long as the longest side. What is the degree measure of the largest angle of this quadrilateral?

8. (UMD 2007) Equilateral triangle AEB lies on top of square $ABCD$. Compute $\angle CED$.

9. (AMC 10 2009) The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x?



10. (ARML 2006) $ABCDE$ is a regular pentagon and $CDMN$ is a square with M, N on the interior of $ABCDE$. Compute $\angle AMN - \angle EAM$.

11. (MAΘ 1992) In regular n-gon $A_1A_2 \ldots A_n$, we have $\angle A_1A_3A_4 = 120°$. Compute n.

4 Challenge Problems

1. (AMC 10B 2008) Quadrilateral $ABCD$ has $AB = BC = CD$, $\angle ABC = 70°$, and $\angle BCD = 170°$. What is the degree measure of $\angle BAD$?

2. (UMD 2016) Consider an isosceles triangle ABC with $\angle ABC = \angle ACB = 40°$. Let D be a point on line AB such that $|AD| = |BC|$ and B lies between A and D. Compute $\angle BCD$.

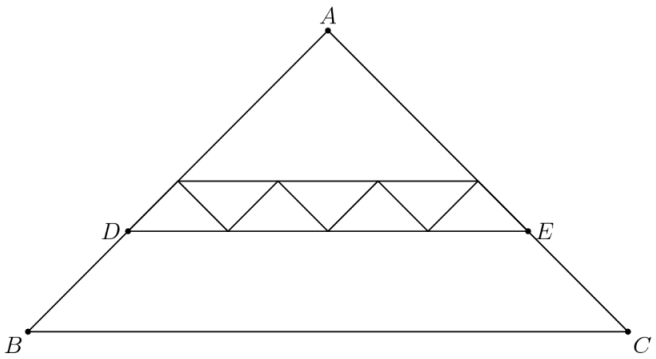
3. (AMC 10 2010) Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

Varsity Practice

David Altizio

5 Warm-up Questions

1. [AMC 10A 2018] All of the triangles in the diagram below are similar to iscoceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



2. [Wikipedia, et. al.] A closed planar shape is said to be *equiable* if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that for any closed shape $\mathcal{T}$ there exists some (nontrivial!) closed shape $\mathcal{T}'$ such that $\mathcal{T}' \sim \mathcal{T}$ and $\mathcal{T}'$ is equible.

3. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Square $BCYX$ is erected outside $\triangle ABC$. Segment $\overline{AX}$ intersects $\overline{BC}$ at point P, while $\overline{AY}$ intersects it at point Q. Determine the length of $\overline{PQ}$.

4. [OMO 2014] The points A, B, C, D, E lie on a line ℓ in this order. Suppose T is a point not on ℓ such that $\angle BTC = \angle DTE$, and $\overline{AT}$ is tangent to the circumcircle of triangle BTE. If $AB = 2$, $BC = 36$, and $CD = 15$, compute DE.

6 Problems

Note: Some of the problems toward the end of this set may creep into topics that we will be covering in future lectures.

1. [CMIMC 2017, David Altizio] Let ABC be a triangle with $\angle BAC = 117°$. The angle bisector of $\angle ABC$ intersects side AC at D. Suppose $\triangle ABD \sim \triangle ACB$. Compute the measure of $\angle ABC$, in degrees.

2. [HMMT Geometry 2002] Let $\triangle ABC$ be equilateral, and let D, E, and F be points on sides BC, CA, AB respectively, with $FA = 9$, $AE = EC = 6$, $CD = 4$. Determine the measure (in degrees) of $\angle DEF$.

3. [AHSME 1986] In $\triangle ABC$, $AB = 8$, $BC = 7$, $CA = 6$ and side BC is extended, as shown in the figure, to a point P so that $\triangle PAB$ is similar to $\triangle PCA$. What is the length of PC?

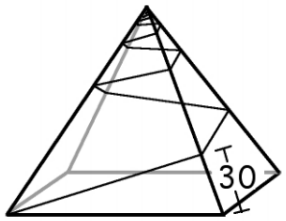
4. [CMIMC 2018, David Altizio] Let ABC be a triangle with side lengths 5, $4\sqrt{2}$, and 7. What is the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$?

5. [Mandelbrot 2006-2007] Suppose that $ABCD$ is a trapezoid in which $\overline{AD} \parallel \overline{BC}$. Given $\overline{AC} \perp \overline{CD}$, $\overline{AC}$ bisects angle $\angle BAD$, and $area(ABCD) = 42$, then compute $area(ACD)$.

6. Let $\mathcal{T}$ be a right triangle with side lengths 3, 4, and 5.
   (a) [AMC 10A 2017] A square with side length x is inscribed in $\mathcal{T}$ so that one vertex of the square coincides with the right-angle vertex of the triangle. What is x?
   (b) [AMC 10A 2017, AMC 10B 2007] A square with side length y is inscribed in $\mathcal{T}$ so that one side of the square lies on the hypotenuse of the triangle. What is y?

7. [AHSME 1981] In $\triangle ABC$, M is the midpoint of side BC, AN bisects $\angle BAC$, and $BN \perp AN$. If sides AB and AC have lengths 14 and 19, respectively, then find MN.

8. [Mandelbrot 2008-2009] A pyramid has a square base, triangular sides, and eight edges that are each 80 meters long. A straight path begins at one corner of the square base, slanting upwards to meet the next edge at a point 30 meters along that edge from the corner, as shown. The path continues around the pyramid, always slanting upward at the same angle, making infinitely many turns. What is the total length of the path?



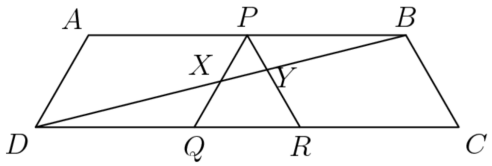
9. [AIME 2015] In the diagram below, $ABCD$ is a square. Point E is the midpoint of $\overline{AD}$. Points F and G lie on $\overline{CE}$, and H and J lie on $\overline{AB}$ and $\overline{BC}$, respectively, so that $FGHJ$ is a square. Points K and L lie on $\overline{GH}$, and M and N lie on $\overline{AD}$ and $\overline{AB}$, respectively, so that $KLMN$ is a square. The area of $KLMN$ is 99. Find the area of $FGHJ$.

10. [AIME 1998] Let $ABCD$ be a parallelogram. Extend $\overline{DA}$ through A to a point P, and let $\overline{PC}$ meet $\overline{AB}$ at Q and $\overline{DB}$ at R. Given that $PQ = 735$ and $QR = 112$, find RC.

11. [Thomas Mildorf] ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15, DE = 2$, and $BC = 16$, then compute CD.

12. [Math League HS 1977-1978] In $\triangle ABC$, $AC = 18$, and D is the point on $\overline{AC}$ for which $AD = 5$. Perpendiculars drawn from D to $\overline{AB}$ and $\overline{BC}$ have lengths 4 and 5 respectively. What is the area of $\triangle ABC$?

13. [Mandelbrot] Figure $ABCD$ below has sides $AB = 6, CD = 8, BC = DA = 2$, and $AB \parallel CD$. Segments are drawn from the midpoint P of AB to points Q and R on side CD so that PQ and PR are parallel to AD and BC as shown. Diagonal DB intersects PQ at X and PR at Y. Evaluate PX/YR.



7 Challenge Problems

14. [Stanford 2012] In quadrilateral $ABCD$, $\angle ABD = \angle BCD$ and $\angle ADB = \angle ABD + \angle BDC$. If $AB = 8$ and $AD = 5$, find BC.

15. [AIME 2003] In $\triangle ABC$, $AB = 360$, $BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. Find the ratio $DE : EF$.