Pigeonhole Principle

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1 JV

2 Warm Up

1. Suppose I have eight different pieces of candy that I need to give to three children.
   (a) In how many ways can I distribute the candy to the children?
   (b) In how many ways can I distribute the candy if every child must receive at least one piece of candy?

2. There are 60 people in an elementary school class. Of the 60 people, 15 of them play basketball, 20 of them play chess, and 24 of them play piano. Furthermore, 12 of them both play basketball and read piano, 10 of them play both basketball and chess, and 8 of them play both chess and piano. 13 of them play only piano. How many people:
   (a) Play all of basketball, piano, and chess?
   (b) Play none of basketball, piano, and chess?

3 Problems

1. For each of the following, how many are there?
   (a) Number of phone numbers.
   (b) Number of ten-digit numbers.
   (c) Number of ten-digit numbers that are divisible by 3.
   (d) Number of ten-digit numbers that are palindromes.
   (e) Number of ten-digit numbers where no two adjacent digits are equal.

2. Suppose five people Annie, Bob, Cassie, David, and Englehart are sitting in the front row of a movie theatre. How many ways are there to arrange them, assuming that Bob and Cassie must sit together?

3. (BmMT 2013 Problem 12) How many three digit even numbers are there with an even number of even digits?

4. Jones has a collection of seven books, and he wishes to take the top three books of this collection and rank them. How many possible rankings are there, assuming that *Moby Dick* is among the top three?

5. (2005 AMC 10A Problem 14) How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

6. How many ways can you rearrange the letters of the word "PITTSBURGH" such that the two T's are not next to each other?

7. Stan is at the lower-left corner of a 7×7 set of city blocks. Call his position $(0,0)$. He needs to get to the point $(7,7)$ while moving only up and right. To make matters more complicated, he needs to avoid the points $(1,1),(2,2)$ as the traffic at those intersections is really bad. How many paths can Stan take?

8. (BmMT 2013 Problem 8) Out of 100 customers at a market, 80 purchased oranges, 60 purchased apples, and 70 purchased bananas. What is the least possible number of customers who bought all three items?

9. (BmMT 2014 Problem 11) Alice, Bob, Clara, David, Eve, Fred, Greg, Harriet, and Isaac are on a committee. They need to split into three subcommittees of three people each. If no subcommittee can be all male or all female, how many ways are there to do this? (4 female, 5 male)

10. In an 20×10 grid, how many rectangles can be formed with corners on the grid and edges parallel to the sides?

4 Challenge Problems

1. (Purple Comet 2010) Ten distinct points are placed in a circle. All ten of the points are paired so that the line segments connecting the pairs do not intersect. In how many different ways can this pairing be done?

2. (Purple Comet 2015) How many subsets of $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ have the property that no two of its elements differ by more than 5? For example, count the sets $\{3\}$, $\{2,5,7\}$, and $\{5,6,7,8,9\}$ but not the set $\{1,3,5,7\}$.

   Solution. Any subset of $[6]$ is permissible, there are $2^6 = 64$ of those. For each $k = 7,\ldots,12$, we count the number of sets with k as their largest number. The only other permissible numbers are $k-5, k-4, k-3, k-2, k-1$, and any subset of these numbers gives a permissible set, so there are $2^5 = 32$ such sets. The total is $64 + 6 \cdot 32 = \boxed{256}$.

5 Common Principles

1. **Factorial** The number of ways to arrange n objects is $n! = n \cdot (n-1) \cdot \cdots \cdot 1$. This is because there are n choices for the first object, $n-1$ choices from the remaining objects to be the second object, and so forth.

2. **Binomial Coefficient** Let $n \geq k$ be two nonnegative integers. Then $\binom{n}{k}$ denotes $\frac{n!}{k!(n-k)!}$. This is the number of ways to choose k objects from a set of n distinct objects.

3. **Complementary Counting** Sometimes it's easier to count the number of ways to *not* satisfy a certain property and remove this count from the total count than it is to count all ways to do something; this is often the case when counting something directly would result in many cases.

6 Varsity

7 Warm-Up

1. How many words (strings of letters) can be formed by rearranging the letters of PENNSYLVANIA?

2. I have eight identical pieces of candy to give to three children. In how many distinct ways can I distribute the candy?

   Solution. Stars and bars with 8 stars and $3-1=2$ bars, so the answer is $\binom{10}{2}=45$.

3. How many integers less than 900 are relatively prime to 900?

   Solution. The

8 Problems

1. In an 20×10 grid, how many rectangles can be formed with corners on the grid and edges parallel to the sides?

   Solution. Choose 2 of the 21 horizontal lines as the top/bottom and 2 of the 11 vertical lines as the left/right. Total: $\binom{21}{2}\cdot\binom{11}{2}=\boxed{11,550}$.

2. Your best friend is trying to name their pet bird; they're determined to pick a name with 6 letters which must contain some vowel to maintain readability. How many possible names are there for this bird?

   Solution. Complementary counting. Number of 6-letter strings: 26^6. Number of 6 letter strings with no vowels: 21^6 (if you assume y is a vowel, 20^6). Number of 6-letter strings with at least one vowel: $\boxed{26^6-21^6}$.

3. (Purple Comet 2008) There are three men and eleven women taking a dance class. In how many different ways can each man be paired with a woman partner and then have the eight remaining women be paired into four pairs of two?

   Solution. Pick the partner for the three men in order: $11\cdot 10\cdot 9=990$ ways to do this. For any pairing of the women, we can permute the pairs and switch the order in each pair without changing the pairing. There are $4!\cdot 2^4$ ways to do this, so the the number of ways to pair off the women is $\frac{8!}{4!2^4}=210$. Total of $990\cdot 210=\boxed{207900}$

4. (Purple Comet 2009) How many ordered triples (a,b,c) of odd positive integers satisfy $a+b+c=25$?

   Solution. Write $a=2k+1$, $b=2m+1$, $c=2n+1$. Then the triples (a,b,c) are in bijection with triples (k,m,n) such that $k+m+n=11$. By stars and bars, there are $\binom{13}{2}=\boxed{78}$ triples.

5. (AIME II 2004) Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.

   Solution. Inclusion/Exclusion.

   (a) Number of divisors of $10^{10} = 2^{10} \cdot 5^{10} : (10+1)(10+1) = 121$.
   (b) Number of divisors of $15^7 : (7+1)(7+1) = 64$.
   (c) Number of divisors of $18^{11} = 2^{11}3^{22} : (11+1)(22+1) = 276$.
   (d) GCD of 10^{10} and 15^7 is 5^7, so number of common divisors is $7+1=8$.
   (e) GCD of 10^{10} and 18^{11} is 2^{10}, so number of common divisors is $10+1=11$.
   (f) GCD of 18^{11} and 15^7 is 3^7, so number of common divisors is $7+1=8$.
   (g) GCD of 10^{10}, 15^7, 18^{11} is 1, so number of common divisors is 1.

   By Inclusion/Exclusion, the total number of divisors is $121+64+276-8-11-8+1 = \boxed{435}$.

6. (HMMT 2004) A classroom consists of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and not which student sits at each desk), how many possible arrangements are there? **Solution.** Pick at most 4 rows and at most 4 columns: $31^2 = 961$. Pick 5 rows or columns: every seat filled so only 1 way. Total $\boxed{962}$.

7. (HMMT 2008) Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? Assume that all animals are distinguishable from each other.

   Solution. 7 of the 9 cows/pigs must be paired with a horse, so there is exactly one pair of a cow and a pig. 20 ways to choose this pair, then $7! = 5040$ ways to pair the remaining cows and pigs to the horses. Total is $20 \cdot 5040 = \boxed{100800}$.

8. (Purple Comet 2015) How many subsets of $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ have the property that no two of its elements differ by more than 5? For example, count the sets $\{3\}$, $\{2,5,7\}$, and $\{5,6,7,8,9\}$ but not the set $\{1,3,5,7\}$.

   Solution. Any subset of [6] is permissible, there are $2^6 = 64$ of those. For each $k = 7, \ldots, 12$, we count the number of sets with k as their largest number. The only other permissible numbers are $k-5, k-4, k-3, k-2, k-1$, and any subset of these numbers gives a permissible set, so there are $2^5 = 32$ such sets. The total is $64 + 6 \cdot 32 = \boxed{256}$.

9. (AIME 1986) In a sequence of coin tosses, one can keep a record of instances in which a tail is immediately followed by a head, a head is immediately followed by a head, and etc. We denote these by TH, HH, and etc. For example, in the sequence TTTHHTHTTTHHTTH of 15 coin tosses we observe that there are two HH, three HT, four TH, and five TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH, and five TT subsequences?

   Solution. Key observation: the subsequences go $TT \to TH \to HH \to HT \to TT$, with possible repetitions of HH and TT. Since there are three HT and four TH, the sequence can be built from $THTHTHTH$ by inserting 1 H and five T next to that same letter.

   There are 4 ways to place the H and $\binom{8}{5} = 56$ ways to place the Ts for a total of $4 \cdot 56 = \boxed{224}$ sequences.

10. (cf. Purple Comet 2009) Five indistinguishable black marbles and seven indistinguishable white marbles are arranged in a line. How many arrangements are there such that every black marble is next to a white marble?

    Solution. Imagine the white marbles in fixed positions, forming 8 slots between them (including the two on either end) for the black marbles to go in. At most one black marble can go into the end slots and at most two into the middle slots. There are three cases:

    (a) All marbles go into different slots: $\binom{8}{5} = 56$ possibilities.

    (b) One pair of marbles goes in one slot, and the rest go in different slots: 6 possibilities for which slot to put the pair in, and $\binom{7}{3} = 35$ possibilities for the last 3 marbles, total of $6 \cdot 35 = 210$.

    (c) Two pairs of marbles each go into one slot, and the other marble in a different slot: $\binom{6}{2} = 15$ possibilities for the two pairs and 6 for the last marble, total of $6 \cdot 15 = 90$.

    The total is $56 + 210 + 90 = \boxed{356}$.

9 Challenge Problems

1. A triangle on the surface of a sphere of radius 1 has angles 60°, 70°, and 80°. Compute its area.

2. (Putnam 2005) For positive integers m, n let $f(m, n)$ denote the number of n-tuples $(x_1, \ldots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \le m$. Show that $f(m, n) = f(n, m)$.

10 Common Principles

- Stars and Bars/Balls and Urns: The number of ways to arrange n stars and k bars or equivalently the number of ways to put n balls into $k + 1$ urns is $\binom{n+k}{n}$. It is important to remember that when doing this sort of count with "dividers", the areas outside the dividers also count as valid groups for your objects!

- Inclusion/Exclusion: If you categorize the objects you're counting in various overlapping sets $A_1, \ldots, A_n$, Inclusion/Exclusion says that you can find the total number of objects by the alternating sum $\sum_{i \in n} |A_i| - \sum_{i<j\in[n]} |A_i \cap A_j| + \sum_{i<j<k\in[n]} |A_i \cap A_j \cap A_k| - \ldots$ which corrects for over/undercounting by considering the intersections of the sets