Algebra Review 2

Tudor Dimitre-Popescu

1 JV Warm-Up

1. Find the unique real number α such that $x^3 - 4x^2 + \alpha x + 18 = 0$ has eactly two solutions in the real numbers.

   Solution. The real solution must be a double root, so the solutions must be r, r, s, with $r, s \in \mathbb{R}$. By Vieta we have that: $2r + s = 4, 2rs + r^2 = \alpha, r^2 s = -18$. Substitute s to get that $r^3 - 2r^2 - 9 = 0$.

2. Find the sum of all roots of $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$.

   Solution. By Vieta, we want $s_1 = -\frac{a_{n-1}}{a_n}$. Apply binomial theorem to get $s_1 = 500$.

3. Let x_1, x_2, x_3 be the roots of $x^3 - 3x - 1 = 0$. Compute $\frac{1}{x_1-2} + \frac{1}{x_2-2} + \frac{1}{x_3-2}$.

   Solution. Expanding, the desired sum is just $\frac{12-4(x_1+x_2+x_3)+(x_1x_2+x_2x_3+x_3x_1)}{8-4(x_1+x_2+x_3)+2(x_1x_2+x_2x_3+x_3x_1)-x_1x_2x_3} = 9$, by Vieta.

2 Problems

1. (AIME 2009) Call a 3-digit number geometric if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

2. (AIME 2014) Let $x_1 < x_2 < x_3$ be three real roots of equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

3. (AMC12 2008 A) The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12^{th} term of the sequence is $\log b^n$. What is n?

4. (AIME 2018) For each ordered pair of real numbers (x, y) satisfying

   $$\log_2(2x + y) = \log_4(x^2 + xy + 7y^2)$$

   there is a real number K such that

   $$\log_3(3x + y) = \log_9(3x^2 + 4xy + Ky^2).$$

   Find the product of all possible values of K.

5. (AIME 2016) For $-1 < r < 1$, let $S(r)$ denote the sum of the geometric series

   $$12 + 12r + 12r^2 + 12r^3 + \dots.$$

   Let a between -1 and 1 satisfy $S(a)S(-a) = 2016$. Find $S(a) + S(-a)$.

6. (AIME 2016) The sequences of positive integers $1, a_2, a_3, \ldots$ and $1, b_2, b_3, \ldots$ are an increasing arithmetic sequence and an increasing geometric sequence, respectively. Let $c_n = a_n + b_n$. There is an integer k such that $c_{k-1} = 100$ and $c_{k+1} = 1000$. Find c_k.

7. (AIME 2015) Let $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$. For a given a, P has exactly two values for c for which the roots are all positive integers. Find the sum of all possible value of c.

3 Varsity Warm-Up

1. Find the unique real number α such that $x^3 - 4x^2 + \alpha x + 18 = 0$ has eactly two solutions in the real numbers.

   Solution. The real solution must be a double root, so the solutions must be r, r, s, with $r, s \in \mathbb{R}$. By Vieta we have that: $2r + s = 4, 2rs + r^2 = \alpha, r^2 s = -18$. Substitute s to get that $r^3 - 2r^2 - 9 = 0$.

2. Find the sum of all roots of $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$

   Solution. By Vieta, we want $s_1 = -\frac{a_{n-1}}{a_n}$. Apply binomial theorem to get $s_1 = 500$.

3. Let x_1, x_2, x_3 be the roots of $x^3 - 3x - 1 = 0$. Compute $\frac{1}{x_1 - 2} + \frac{1}{x_2 - 2} + \frac{1}{x_3 - 2}$.

   Solution. Expanding, the desired sum is just $\frac{12 - 4(x_1+x_2+x_3) + (x_1x_2 + x_2x_3 + x_3x_1)}{8 - 4(x_1+x_2+x_3) + 2(x_1x_2+x_2x_3+x_3x_1) - x_1x_2x_3} = 9$, by Vieta.

4 Problems

1. (AMC 2008 A) A sequence (a_1, b_1), (a_2, b_2), (a_3, b_3), $\ldots$ of points in the coordinate plane satisfies

   $$(a_{n+1}, b_{n+1}) = (\sqrt{3}a_n - b_n, \sqrt{3}b_n + a_n) \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

   Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

2. If $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$, prove that $\frac{x^6 + y^6 + z^6}{x^3 + y^3 + z^3} = xyz$. **Solution.** Outline: Multiply $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ by xyz to get that $xy + xz + yz = 0$, so x, y, z are the solutions of $t^3 - k = 0$, for some constant k. Then $x^3 = y^3 = z^3 = k$, and by Vieta we are done.

3. (HMMT 2009) If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

4. (HMMT 2009) Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

5. (AMC 2006) Let $a_1, a_2, ...$ be a sequence for which

   $$a_1 = 2 \qquad a_2 = 3 \qquad \text{and} \qquad a_n = \frac{a_{n-1}}{a_{n-2}} \text{ for each positive integer } n \geq 3.$$

   What is a_{2006}?

6. (Putnam 2016)* Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.