Expected Value and Conditional Probability

Annie Xu and Ariel Uy

January 21, 2018

1 Warm-Up

1. What is the expected value of rolling a standard 6-sided die?
2. Annie is playing basketball. She can make 2-point shots 10% of the time and 3-points shots 5% of the time. What is the expected value of her score when she attempts a 2-point shot? Which one should she be attempting (has the higher expected value)?
3. 2 standard 6-sided dice are rolled, and the sum of their faces is 10. What is the probability that the first dice shows a 5?
4. You flip a coin 2 times. What is the probability that you get 2 heads? What is the probability that you get 2 heads given that the first coin was a head? What is the probability you get 2 heads given you see at least 1 head?

2 Expected Value

1. Misha is out gathering nuts for his squirrels. 1 out of every 40 nuts he gathers is rotten inside, but unfortunately Misha cannot tell which nuts are rotten until the squirrel throws it at his head. For every good nut, Misha is given $3 in gratitude by the squirrel, and for every bad nut, he loses $30 in medical fees. Does Misha lose or gain money in the long-term?
2. You have a fair coin, and flip it until you get heads. What is the expected number of flips it will take you?
3. You have a fair coin, and flip it until you get the exact sequence HHT. What is the expected number of flips it will take you?
4. You have an unfair coin, which is heads with probability 0.6, and flip it 10 times. What is the expected number of TT sequences? (TTT has 2 TT sequences.)
5. 5 people are attending a party, and each has their own nametag. Before they enter, all their name tags are randomly shuffled and handed out to them. What is the expected number of people with their correct name tag? What happens if there are n people at the party?
6. You have a bag with n pieces of string in it. You reach into the bag and randomly grab 2 ends - they may be from the same or different strings. You tie these ends together and place the string back in the bag. Repeat this process until there are no more untied ends. What is the expected number of loops?

3 Conditional Probability

1. A bag contains 5 coins, 3 of which are normal (heads-tails) and 2 of which are rigged (heads-heads). Picking a coin at random from the bag and flipping it, what is the probability you get heads? If you see that you get heads, what is the probability the coin you picked is rigged?

2. David runs to deliver a message to overlord Misha every day. He is either on-time or late, and then is either yelled at or not. He is late 60% of the time. If he is late, Misha yells at him with a probability of 90%. If he is on-time, Misha just yells at him anyway 20% of the time. You enter Misha's overlord lair, and hear him yelling at David. What is the probability that David was late?

3. One week, your friend randomly chooses a day to go to the store and buys a piece of fruit, randomly choosing either an apple or a banana. The next week, she does the same thing. Assume she owns no other fruit and is equally likely to buy fruit on any day of the week.

   (a) Given that one of the fruits is an apple, what is the probability that your friend has no bananas?

   (b) Given that one of the fruits is an apple that was bought on a Wednesday, what is the probability that your friend has no bananas?

4. (Math Prize 2009) Consider a fair coin and a fair 6-sided die. The die begins with the number 1 face up. A step starts with a toss of the coin: if the coin comes out heads, we roll the die; otherwise (if the coin comes out tails), we do nothing else in this step. After 5 such steps, what is the probability that the number 1 is face up on the die?

4 Challenge Problem

3 people play the following game. Each person is receives either a white or black hat without seeing its color. The hat colors are chosen independently and uniformly at random. Once everyone has a hat, they can see each others' hats, but cannot communicate in any way. Then everyone must simultaneously either guess the color of their hat or say "no guess." In order to win, they need at least one person to correctly guess their hat color, and no one to guess incorrectly. If someone guesses incorrectly, or no one guesses, then they lose. Before they are given hats, they convene to strategize.

(a) Find a strategy that maximizes the probability that they win.

(b) Prove that the strategy you gave in part (a) is optimal. That is, show that any strategy has at most the same probability of winning.

(c) This game can be played with n players, where $n \geq 1$ is any natural number. Give an upper bound on the probability of winning that is a simple function of n (there is a proof for part (b) that generalizes easily to give this bound).

(d) Prove that if this upper bound is attainable then $n = 2^k - 1$ for some natural number k.

(e) Challenge: Prove that if $n = 2^k - 1$, then this bound is attainable.

5 Background

Definition 1. The expected value of a random variable X, or $E[X]$, is the average value of that variable over a long period of time. For random variables with discrete values, this is the weighted average of all the outcomes. $E[X] = \sum_{events} Pr[event] \times Value[event]$

Theorem 2. Let the random variable X be composed of the sum of n random variables X_i, so $X = \sum_{i=1}^{n} X_i$. Note that these events do not need to be independent. Linearity of Expectation states that the expected value of X is linear, or that $E[X] = \sum_{i=1}^{n} E[X_i]$.

Definition 3. The probability of event A given event B, or $Pr[A \mid B]$, is the probability that event A happens given event B happens. In other words, it is the probability that both A and B occur divided by the probability that B occurs. $Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$

Theorem 4. Given 2 events A and B, we can rearrange the definition of conditional probability to get a very important formula called Bayes' Theorem: $Pr[A \mid B] = \frac{Pr[B|A]Pr[A]}{Pr[B]}$

Theorem 5. The Law of total probability lets you split up a probability based on the cases of event that can occur. If events B and C are disjoint and are the only possible cases that can happen, then to find the probability of some other event A, we can use the fact that: $Pr[A] = Pr[A \mid B]Pr[B] + Pr[A \mid C]Pr[C]$.