Number theory

1. Think about it

1. $2n = p + q$, where p and q are two consequent odd prime numbers. Prove that n is composite.
2. There are three machines that can print pairs of numbers. If you input pair (n, m) to the first machine, it will print out $(n - m, m)$. The second machine: $(n + m, m)$. The third machine (m, n). You are holding pair $(46, 51)$ is your hands. Can you get pair $(15, 33)$ using these three machines? (you can use them as many times as you want, and in any order you want)
3. Find all positive integers (n, m) such that $4^n + 4^m$ is a perfect square.
4. Number 28^{101} is written on the board. Alex counted the sum of digits of this number and write on the board. The CJ counted the sum of Alex's number and wrote the result. Then they did it again with the last number. What is the last number written on the board?
5. Is 999999973 a prime number?

2. Warm-Up

6. $n^2 + 1$ is a 10-digit number. Prove, that it has two equal digits.
7. Find all positive integers such that $n + S(n) = 1000$.
8. Prove that for every positive integer n, fraction $\frac{n^4+4n^2+3}{n^4+6n^2+8}$ is in the lowest term.
9. Is it possible, using digits from 1 to 6 one time each, construct a multiple of 11?
10. p and $p^2 + 2$ are prime. Prove that $p^3 + 2$ also a prime.
11. n is not divisible by 17. Prove that either $(n^8 - 1) \vdots 17$ or $(n^8 + 1) \vdots 17$.
12. Compute $\left[\frac{2^0}{3}\right] + \ldots + \left[\frac{2^{1000}}{3}\right]$.

3. Problems

13. Find all integers n such that $1 + n + n^2 + n^3$ is a power of three.
14. Let $a + b + c = (a - b)(b - c)(c - a)$. Prove that $a + b + c \vdots 54$.
15. Prove that there exist infinitely many integers that cannot be represented as a sum of two perfect cubes. Is this statement true for three cubes?
16. Find all primes p, such that $4p + 1$ is a perfect square.
17. Find all primes p, such that $p^2 + 11$ has exactly 6 different positive divisors.
18. Is $42^{47} + 47^{42}$ a prime number?
19. Find the reminder of 2^{98} when divided by 33.

20. Find last two digits of 1032^{1032}.
21. Find last three digits of $2008^{2007^{2006^{\cdots}}}$

4. Bonus

22. Prove that there exist infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natural number n.
23. Prove that $p^q + q^p \equiv p + q \mod pq$, where p, q are two different prime numbers.
24. p is a fixed prime number. Prove that for every c there exists a solution to $x^x \equiv c \mod p$.
25. Call a lattice point "visible" if the greatest common divisor of its coordinates is 1. Prove that there exists a 100×100 square on the board none of whose points are visible.
26. Prove that for all n we can find a set of n consecutive integers such that none of them is a power of a prime number.
27. Let p be a prime. Show that there are infinitely many positive integers n such that p divides $2^n - n$.