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Divisibility	
Western PA ARML Practice	*September 20, 2015*

Warm-up

1. *(ARML 1991) Compute the smallest 3-digit multiple of 7 for which the sum of its digits is also a multiple of 7.*

   Let a, b, c be the digits of the number. Then $100a + 10b + c \equiv a + b + c \equiv 0 \pmod 7$, so $99a + 9b \equiv 0 \pmod 7$, which reduces to $a + 2b \equiv 0 \pmod 7$.

   To be a legitimately 3-digit number, a must be at least 1. The least (and only) value of b such that $1 + 2b \equiv 0 \pmod 7$ is 3, so $b = 3$. To have $a + b + c \equiv 0 \pmod 7$, take $c = 3$ as well, giving the answer 133.

1 The divisors of an integer

1. *(AIME 1998) A divisor of* 10^{99} *is chosen uniformly at random. Find the probability that it's divisible by* 10^{88}.

   A divisor of 10^{99} has the form $2^x \cdot 5^y$, where $0 \le x \le 99$ and $0 \le y \le 99$. It is divisible by 10^{88} if $x \ge 88$ and $y \ge 88$. So 12 of the 100 possibilities for x, and 12 of the 100 possibilities for y, result in a multiple of 10^{88}, which means that the probability is $\left(\frac{12}{100}\right)^2 = \frac{9}{625}$.

2. *Find the number of ways to write* 300 *as a product of three positive integers* $a \cdot b \cdot c$. *(The product is ordered, so* $1 \cdot 3 \cdot 100$ *is different from* $100 \cdot 1 \cdot 3$.*)*

   We have $300 = 2^2 \cdot 3 \cdot 5^2$. For each prime, we must choose how to distribute its factors between a, b, and c.

   For 2^2, we have three ways to order $4 \cdot 1 \cdot 1$ and three ways to reorder $2 \cdot 2 \cdot 1$: six possibilities. The same happens for 5^2. For 3, we have three possibilities: $3 \cdot 1 \cdot 1$, $1 \cdot 3 \cdot 1$, or $1 \cdot 1 \cdot 3$. So the total number of possibilities is $3 \cdot 6^2 = 108$.

3. *Call* n *an* everyday number *if the sum of the divisors of* n *(including* n *itself) is even. For example, 6 is an everyday number, since* $1+2+3+6 = 12$, *but 8 is not, since* $1+2+4+8 = 15$. *How many of the divisors of* 10^{100} *are everyday numbers?*

   If $n = 2^x \cdot 5^y$, then the sum of the divisors of n is $(1 + 2 + 4 + \cdots + 2^x) \cdot (1 + 5 + 25 + \cdots + 5^y)$. The first factor is always odd, so it won't affect the everydayness of n; the second factor adds up $y + 1$ odd numbers, so it's even whenever $y + 1$ is even—when y is odd.

   There are 101 choices for x, if $0 \le x \le 100$, and 50 choices for y, if $0 \le y \le 100$ and y is odd. Therefore $101 \cdot 50 = 5050$ divisors of 10^{100} are everyday numbers.

4. *(Well-known) Suppose you're in a hallway with 100 closed lockers in a row, and 100 students walk by. The first student opens every locker. The second student closes every other locker. The third student goes to every third locker and toggles it: opens it if it's closed, and closes it if it's open. The remaining students continue this process: the n-th student goes to every n-th locker and toggles it. When all 100 students have walked by, which lockers are open?*

   The trick is to reverse the description: if the n-th student toggles lockers which are a multiple of n, then the n-th locker is toggled by students which are a divisor of n. A locker ends up open if it's been toggled an odd number of times, so we want to know the numbers between 1 and 100 with an odd number of divisors.

   Unless n is a perfect square, each divisor d of n can be paired with another divisor $\frac{n}{d}$, making an even number of divisors. (If n is a perfect square, then $\sqrt{n}$ is left over.) So the perfect squares—lockers 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100—are the only ones left open.

   (Alternatively: if the prime factorization of n is $p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then n has $(a_1+1)(a_2+1)(\cdots)(a_k+1)$ divisors, which is odd only if every a_i is even: when n is a perfect square.)

5. *(ARML 1984) Find all possible values of k for which $1984 \cdot k$ has exactly 21 positive divisors.*

   A number n has 21 positive divisors if it's of the form $p^2 \cdot q^6$ (where p and q are primes) or p^{20} (where p is prime). Since 1984 factors as $2^6 \cdot 31$, the only way to put $1984 \cdot k$ into this form is to make $k = 31$, so we get $1984 \cdot 31 = 2^6 \cdot 31^2$.

6. *Let n be of the form $2^a \cdot 3^b$ for some a and b. Prove that the sum of the divisors of n (including n itself) is at most $3n$.*

   The sum of the divisors of n is given by $(1+2+2^2+\cdots+2^a)(1+3+3^2+\cdots+3^b)$. The inequalities

   $$\begin{cases} 1+2+2^2+\cdots+2^a = 2^{a+1}-1 < 2^{a+1} \\ 1+3+3^2+\cdots+3^b = \frac{1}{2}(3^{b+1}-1) < \frac{1}{2}\cdot 3^{b+1} \end{cases}$$

   together imply that the sum of the divisors of n is less than $2^{a+1}\cdot\frac{1}{2}\cdot 3^{b+1} = 2^a \cdot 3^{b+1} = 3n$.

7. *(PUMaC 2011) The sum of the divisors of n (including n itself) is 1815. If $n = 2^a \cdot 3^b$ for some a and b, find (a, b).*

   We know that 1815, which factors as $3\cdot 5\cdot 11^2$, is equal to $(1+2+2^2+\cdots+2^a)(1+3+3^2+\cdots+3^b)$. For the first few values of a, the first factor is $1, 3, 7, 15, 31, \dots$ and for the first few values of b, the second factor is $1, 4, 13, 40, 121, 364, \dots$. We spot (by looking for factors of 11, which are rare) that $1815 = 15 \cdot 121$, so $n = 2^3 \cdot 3^4 = 8 \cdot 81 = 648$.

8. *(ARML 1979) Let $\tau(n)$ denote the number of positive divisors of n. (E.g., $\tau(12) = 6$, counting $1, 2, 3, 4, 6$, and 12 itself.) For how many positive integers $n \le 100$ is $\tau(n)$ a multiple of 3?*

   If $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then it has $(a_1+1)(a_2+1)(\cdots)(a_k+1)$ divisors, which is a multiple of 3 whenever $a_i + 1$ is a multiple of 3 for some i.

   This happens whenever n is divisible by p^2 and not p^3 for some p, and also when n is divisible by p^5 and not p^6, and also higher powers that are irrelevant for $n \le 100$. We count:

   (a) Numbers divisible by 2^2 and not 2^3 are $4, 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100$.

(b) Multiples divisible by 3^2 and not 3^3 are $9, 18, 36, 45, 63, 72, 90, 99$ (but 36 was already counted).

(c) Numbers divisible by 5^2 and not 5^3 are $25, 50, 75, 100$ (but 100 was already counted).

(d) Numbers divisible by 7^2 and not 7^3 are 49 and 98.

(e) Numbers divisible by 2^5 and not 2^6 are 32 and 96.

Altogether, there are 27 such numbers.

9. *(ARML 2014) Find the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.*

We have $214 = 2 \cdot 107$ and $2014 = 2 \cdot 19 \cdot 53$. If $n = 2^a \cdot 19^b \cdot 53^c \cdot 107^d$, then $214 \cdot n$ has $(a+2)(b+1)(c+1)(d+2)$ divisors and $2014 \cdot n$ has $(a+2)(b+2)(c+2)(d+1)$ divisors. So we want $(b+1)(c+1)(d+2) = (b+2)(c+2)(d+1)$.

After some experimentation, we realize that one way to do this is to set $n = 19^2 \cdot 53$, in which case $b = 2$, $c = 1$, $d = 0$, and $(b+1)(c+1)(d+2) = (b+2)(c+2)(d+1) = 12$. We can check that we can't do better by seeing that $n = 19^3$ doesn't work and that all 7 possibilities with $b + c + d \leq 2$ don't work. Note that introducing new primes can never help us, since it changes the number of divisors of $214 \cdot n$ and of $2014 \cdot n$ by the same factor.

So $n = 19^2 \cdot 53 = 19133$ is the best solution.

2 Prime factorization

1. *Prove that* $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = a \cdot b$.

For every prime p, suppose that p divides a exactly x times, and p divides b exactly y times. (This is denoted $p^x \| a$, $p^y \| b$.) Then $p^{\min(x,y)} \| \gcd(a, b)$ and $p^{\max(x,y)} \| \operatorname{lcm}(a, b)$. Therefore the power of p dividing the right-hand side is $\min(x, y) + \max(x, y) = x + y$, same as the right-hand side.

Since the power of p dividing both sides is the same for any prime p, the two sides must be equal.

2. *(USAMO 1972) Prove that for all positive integers a, b, c,*

$$\frac{\gcd(a, b, c)^2}{\gcd(a, b) \cdot \gcd(a, c) \cdot \gcd(b, c)} = \frac{\operatorname{lcm}(a, b, c)^2}{\operatorname{lcm}(a, b) \cdot \operatorname{lcm}(a, c) \cdot \operatorname{lcm}(b, c)}.$$

Once again, let p be a prime, and let $p^x \| a$, $p^y \| b$, $p^z \| c$. Since the equations are symmetric in a, b, and c, we can assume without loss of generality that $x \leq y \leq z$. Then the power of p dividing the left-hand side is $2x - (x + x + y) = -y$, and the power of p dividing the right-hand side is $(2z - (y + z + z) = -y$.

Since the power of p dividing both sides is the same for any prime p, the two sides must be equal.

3. *(AIME 1991) How many reduced fractions $\frac{a}{b}$ are there such that $ab = 20!$ and $0 < \frac{a}{b} < 1$?*

   The prime factorization of 20! is $2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. If the same prime number appears in the factorization of a and b, then $\frac{a}{b}$ is not reduced, which is not allowed. So each of the eight prime powers that 20! factors into must end up entirely in a or else entirely in b. There are 2^8 ways to do this.

   Exactly half of these will satisfy $\frac{a}{b} < 1$, since either $\frac{a}{b} < 1$ or else $\frac{b}{a} < 1$. So there are $2^7 = 128$ solutions.

4. *Find all solutions to $x^2 + 3x = y^2$, where x and y are positive integers.*

   For any prime p, the power of p dividing y^2 is even, so the power of p dividing $x^2+3x = x(x+3)$ is even. If p divides both x and $x+3$, then $p \mid (x+3) - x = 3$. So for primes p other than 3, the even power of p dividing y^2 must entirely divide either x or $x+3$ as well.

   If the powers of 3 dividing x and $x+3$ also happen to be even (including 0), then x and $x+3$ are perfect squares, which is only possible when $x = 1$ and $x+3 = 4$. If the powers of 3 dividing x and $x+3$ are both odd, then $x/3$ and $(x+3)/3 = x/3+1$ are perfect squares, which is only possible when $x/3 = 0$ and $x/3+1 = 1$ (but this is ruled out because $x > 0$. So the only solution is $x = 1$, which means $y = 2$.

5. *(Putnam 2003) Show that for each positive integer n,*

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\left(1, 2, \ldots, \lfloor n/i \rfloor\right).$$

   For every prime p, the number of times p divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.$$

   This can be seen as the number of points with positive integer coordinates under the graph of the curve $y = n \cdot p^{-x}$. The first term, $\left\lfloor \frac{n}{p} \right\rfloor$, gives the number of such points with $x = 1$. The second term counts points with $x = 2$, and so on.

   We can count these points in a different way as well. If we fix the y-coordinate, the number of points with $y = i$ is given by the largest value of x such that $n \cdot p^{-x} > i$, or $p^x < \frac{n}{i}$. Coincidentally, this happens to equal the largest power of p that divides $\operatorname{lcm}\left(1, 2, \ldots, \lfloor n/i \rfloor\right)$.

   Since the power of p dividing both sides is the same for any prime p, the two sides must be equal.