Combinatorics	Victor Xu
Counting Strategies	
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1 Introduction

What is combinatorics? In regard to math competitions, combinatorics is how to count things. While counting may sound like a simple task, without the proper strategies, it can be quite difficult. If you are not convinced, try the problem below. Combinatorics is closely related to probability on these competitions.

Hard Problem

1. You have marbles of eight different colors, and five brown boxes. How many distinct ways are there to put the marbles in the boxes?

2 Permutations

A permutation of some elements is a rearrangement of those elements. For example, $(2, d, 1)$ is a permutation of $(d, 1, 2)$. A frequent problem is to count the number of permutations of some elements. The simplest case is when all of our n elements are distinct. In this case, we have n choices for the first element, $n - 1$ choices for the second, and so on. Thus, the number of permutations is $n!$. However, if we are unlucky, we might have duplicates in our n elements. In this case, there are still $n!$ ways to rearrange these elements, but some of these rearrangements give the same result. We need to divide by the number of duplicates every distinct arrangement produces. Fortunately, we can do this because the number of duplicates produced is itself a product over every duplicated element of the number of permutations of that duplicated element. Some examples will make this more clear.

Problems

1. Five runners run a race. How many different ways can they finish?
2. How many distinct ways are there to rearrange the letters in REARRANGE?

3 Binomial Coefficients

Suppose we have 10 different pieces of candy, and we want to choose 3 of them to eat. How can we count the number of ways to do this? We can use a similar technique to the one we used to count the number of permutations with non-distinct elements, that is, we can try to use a naive approach, and then divide by the number of times every combination is duplicated in our counting method. We have 10 candies initially, and then 9 candies to choose from, and then 8 candies to choose from, for a total of $10 * 9 * 8 = 720$ ways to choose 3 candies. However, there are $3 * 2$ ways we could have chosen any set of 3 candies: we could have chosen any one of the 3 first, and then any one of the two remaining candies second. So, the number of ways to choose 3 candies is actually $720/3 = 240$.

This example should make the following formula clear. Try to justify why this formula will give the number of ways to choose r elements from n elements.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1}$$

These quantities are called binomial coefficients, because if we expand a power of a binomial, these will be the coefficients of the terms.

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} \tag{2}$$

Problem

1. Five runners run a race. How many different ways can the top three finishers be selected, if we do not care about the specific order of these top three?

4 Stars and Bars

Suppose we have 10 bars of gold to distribute among 5 people. We can count the number of ways to do this with a technique called stars and bars: treat every gold bar as a star, and then have 4 separators which will serve to determine how many gold bars each person gets. If we find the number of distinct rearrangements of these 10 bars of gold and 4 dividers, we can count the number of ways to distribute the gold bars.

Problem

1. Ten identical robots run a race with three different finish lines. How many different ways can the robots be distributed among the finishing lines?

5 Problems

Some of the following problems will use techniques not covered today.

1. In order to list the numbers 0 to 7 in binary notation, we need 12 1's (000, 001, 010, 011, 100, 101, 110, 111). How many 1's are needed to list in binary the numbers from 0 to 1023?
2. Try to justify/prove to yourself the validity of the formulas given.
3. There is a bag with 50 red balls, 50 blue balls, and 30 yellow balls. Given that after pulling out 65 balls at random there are 5 more red balls then blue balls, what is the probability that the next ball pulled out is red?