Series Practice

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Problems

1. (1980 ARML Team Round Problem 6) In an arithmetic progression, the ratio of the sum of the first r terms to the sum of the first s terms is equal to the ratio of r^2 to s^2 ($r \neq s$). Compute the ratio of the 8th term to the 23rd term.

2. (1992 ARML Individual Round Problem 4) Consider the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots,$ where the integer n appears n times. Compute the 1992^{nd} term of this sequence.

3. (1993 ARML Team Round Problem 8) The arithmetic mean of the positive numbers $a_1, a_2, \ldots, a_k$ is one-fourth of the arithmetic mean of the positive numbers $b_1, b_2, \ldots, b_{7k}$ (where k is a positive integer). If both of these means are integers, compute the smallest possible integer value for the arithmetic mean of
$$a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_{7k}$$

4. (1993 ARML Team Round Problem 10) (*Reminder:* The Fibonacci sequence is defined as follows: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.)

   The Fibonacci numbers F_a, F_b, and F_c form an arithmetic sequence. If $a + b + c = 2000$, compute a.

5. (1999 ARML Team Round Problem 7) Define a sequence of integers as follows: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 5$, $a_5 = 7$, $a_6 = 9$, the next four terms are the next four even integers after 9, the next five terms are the next five odd integers after 16, and so on. Compute a_{1999}.

6. (2006 AMC 12A Problem 12) A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the outer rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

7. (2003 AIME I Problem 2) One hundred concentric circles with radii $1, 2, 3, \ldots, 100$ are drawn in a plane. The interior of the circle of radius 1 is colored red, and each region bounded by consecutive circles is colored either red or green, with no two adjacent regions the same color. The ratio of the total area of the green regions to the area of the circle of radius 100 can be expressed as m/n, where m and n are relatively prime positive integers. Find $m + n$.

8. (2012 AIME I Problem 2) The terms of an arithmetic sequence add to 715. Then, a new sequence is constructed by doing the following: The first term of the sequence is increased by 1, the second term is increased by 3, the third term is increased by 5, and in general, the kth term is increased by the kth odd positive integer. The terms of the new sequence add to 836. Find the sum of the first, last, and middle terms of the original sequence.

9. (2005 AIME II Problem 3) An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is m/n where m and n are relatively prime integers. Find $m+n$.

10. (2007 AIME II Problem 12) The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308, \quad 56 \leq \log_3\left(\sum_{n=0}^{7} x_n\right) \leq 57,$$

find $\log_3(x_{14})$.