Generating functions — Misha Lavrov

Generating functions and linear recurrences

Western PA ARML Practice — *March 15, 2015*

1 Useful facts

For any r satisfying $|r| < 1$, the infinite sum $a + ar + ar^2 + ar^3 + \cdots$ converges to:

$$\sum_{k\geq 0} ar^k = \frac{a}{1-r}.$$

In particular, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$.

2 Problems

1. Find the generating functions for each of the following sequences:
   (a) $1, 1, 1, 1, 1, \ldots$ (every term is 1)
   (b) $1, 0, 1, 0, 1, \ldots$ (every even term is 1, and every odd term is 0)
   (c) $1, 3, 9, 27, 81, \ldots$ (the n-th term is 3^n)
   (d) $0, 1, 3, 7, 15, \ldots$ (the n-th term is $2^n - 1$)
2. Find the sequences represented by each of the following generating functions:
   (a) $f(x) = \frac{1}{1+x}$.
   (b) $f(x) = 1$.
   (c) $f(x) = \frac{x^2}{1-2x}$.
   (d) $f(x) = \frac{1}{2-x}$.
   (e) $f(x) = (x+1)^5$.
3. Find the generating functions for each of the following recursively defined sequences:
   (a) $a_n = 2a_{n-1} + 1$, with $a_0 = 0$.
   (b) $b_n = b_{n-1} + 2b_{n-2}$, with $b_0 = 3$ and $b_1 = 6$.
   (c) $c_n = 6c_{n-1} - c_{n-2}$, with $c_0 = 0$ and $c_1 = 1$.
   (d) $d_n = 6d_{n-1} - d_{n-2} - 2$, with $d_0 = 1$ and $d_1 = 2$.
   (e) $e_n = e_{n-1} + e_{n-2} + e_{n-3}$, with $e_0 = e_1 = 0$ and $e_2 = 1$.

4. Solve for the closed form of the sequences in problem 3. (Except possibly for the last one, which requires solving a cubic equation.)

5. Find a formula for the n-th term of the sequence whose generating function is $\frac{x^3}{1-x^3-x^6}$, in terms of n.

6. If the sequence g_n has generating function $G(x) = \frac{1}{1-3x-x^2}$, find a recursive formula expressing g_n in terms of g_{n-1} and g_{n-2}.

7. Without using a calculator or performing long division, compute the first ten digits after the decimal of $\frac{1}{98}$.

8. Prove (you do not need to use generating functions for this) that the sequences defined in problem 3, parts (c) and (d), satisfy $c_n^2 = \binom{d_n}{2}$ for all n.

   (In fact, these two sequences give all the pairs for which such a statement holds.)