Probability & Games

Infinitely long games of chance

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Warm-up

Problem (Review)

Arrange the seven Tetris pieces into a 4×7 *rectangle.*

Problem (Logic Puzzle)

A hundred people are given numbers $1, 2, \ldots, 100$*. Each person rolls a* 100*-sided die, whose sides are also numbered* $1, 2, \ldots, 100$*, and wins if the result is bigger than his or her number (multiple people can win). What is the probability that an odd number of people win?*

Warm-up: Solutions

1. Color the 4×7 square like a checkerboard; then there are 14 black squares and 14 white squares. Each Tetris piece must color an equal number of black and white squares, except for the T-shaped piece (which covers 3 squares of one color and 1 of another). Therefore this task is impossible.

Warm-up: Solutions

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2. Consider person #50, who wins with probability $\frac{1}{2}$. If x other people win, then #50 has a chance of $\frac{1}{2}$ of changing this to $x+1$; exactly one of x and $x+1$ is odd. So the probability that an odd number of people win is $\frac{1}{2}$.

Infinitely long games

Problem (Mathcounts – a long time ago)

Alice, Bob, and Cameron take turns flipping a coin (in alphabetical order). The first to flip heads wins. What is the probability that Alice wins?

Problem (Variant on the above problem)

Suppose Cameron cheats and uses a coin that lands heads with a probability of $\frac{2}{3}$. What is the probability that Alice wins in this version?

Solutions: Infinitely long games

1. If Alice wins with probability p, then Bob wins with probability $\frac{1}{2}p$, and Cameron wins with probability $\frac{1}{4}p$. Therefore $p + \frac{1}{2}p + \frac{1}{4}p = 1$, so $p = \frac{4}{7}$.

Solutions: Infinitely long games

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2. When it's Alice's turn, she either wins with probability $\frac{1}{2}$, or the game comes back around to her with probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{12}$. So the total probability that Alice wins is

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{12} + \frac{1}{2} \cdot \left(\frac{1}{12}\right)^2 + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{12}} = \frac{6}{11}.$$

Infinitely long games with state

Problem (Duke Math Meet 2008/7.)

Two cows begin standing on opposite faces of a cube. They take turns moving to a random adjacent face. If a cow moves to the same face as the other cow, it knocks the other cow off the cube. What is the probability that the first cow to move gets knocked off?

Problem (AIME 1995/15.)

Alice flips a coin over and over. Find the probability that she will encounter a run of 5 heads before encountering a run of 2 tails.

Solution: The one with the cows

The game can be in four essentially different states: two relative positions of the cows (opposite or adjacent) and two cows whose turn it is. Let $p_{\mathrm{opp},1}, p_{\mathrm{opp},2}, p_{\mathrm{adj},1}, p_{\mathrm{adj},2}$ denote the probability that the first cow gets knocked off, in each case. Then we have:

$$\begin{cases} p_{\mathrm{opp},1} &= p_{\mathrm{adj},2} \\ p_{\mathrm{opp},2} &= p_{\mathrm{adj},1} \\ p_{\mathrm{adj},1} &= \frac{1}{4}p_{\mathrm{opp},2} + \frac{1}{2}p_{\mathrm{adj},2} \\ p_{\mathrm{adj},2} &= \frac{1}{4}p_{\mathrm{opp},1} + \frac{1}{2}p_{\mathrm{adj},1} + \frac{1}{4} \end{cases}$$

(There are some tricks we can use to reduce the number of states, but this is left as an exercise.)

Solving, we find that $p_{\mathrm{opp},1} = \frac{3}{5}$.

Solution: AIME 1995/15

This time, we can describe the state by the most recent run of whichever outcome: it can be H, HH, HHH, HHHH, or T. Let p_{state} be the probability of seeing Alice winning, starting in that state. We have

$$\begin{cases} p_{\mathrm{H}} & = \frac{1}{2} p_{\mathrm{T}} + \frac{1}{2} p_{\mathrm{HH}} \\ p_{\mathrm{HH}} & = \frac{1}{2} p_{\mathrm{T}} + \frac{1}{2} p_{\mathrm{HHH}} \\ p_{\mathrm{HHH}} & = \frac{1}{2} p_{\mathrm{T}} + \frac{1}{2} p_{\mathrm{HHHH}} \\ p_{\mathrm{HHHH}} & = \frac{1}{2} p_{\mathrm{T}} + \frac{1}{2} \\ p_{\mathrm{T}} & = \frac{1}{2} p_{\mathrm{H}}. \end{cases}$$

Solving, we find that $p_{\mathrm{H}} = \frac{2}{17}$ and $p_{\mathrm{T}} = \frac{1}{17}$. At the beginning, we go to either of states H or T with probability $\frac{1}{2}$, so the probability is $\frac{3}{34}$.

Even longer infinitely long games

Problem (Mathcounts 2008 Target Round/8.)

Bob flips a coin over and over. His score starts out at 0*, increases by* 1 *with each outcome of heads, and is reset to* 0 *with each outcome of tails.*

If, at any point, Bob's score exceeds the total number of tails flipped, Bob wins the game.

1. *Find the probability that Bob wins with a score of* 4*.*
2. *Prove that with positive probability, the game never ends.*
3. *Find the best estimate you can of the probability that the game goes on forever.*

Solution: Mathcounts 2008 Target Round/8

We can describe the state of the game at any point by a pair (s, t), where s is the score and t the total number of tails. Bob starts in state 0; from state $(0, t)$, if he flips until winning or flipping tails, he has a chance of $2^{-(t+1)}$ of winning, and otherwise ends up in state $(0, t + 1)$.

Solution: Mathcounts 2008 Target Round/8

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1. The probability of the game progressing $(0,0) \to (0,1) \to (0,2) \to (0,3) \to \text{win}$ is $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{7}{8} \cdot \frac{1}{16} = \frac{21}{1024}$.

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2. The probability of winning with score 1 is $\frac{1}{2}$; the probability of winning with score $s \geq 2$ is less than $\frac{1}{2} \cdot 2^{-s}$, which adds to $\frac{1}{4}$ over all $s \geq 2$. So the total probability of winning is $\leq \frac{3}{4}$.

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3. The probability of the game never ending is $\prod_{t=1}^{\infty}(1 - 2^{-t})$ or approximately 0.288788. Best of luck!

Martin Gardner's Truel Problem

Alice, Bob, and Cameron are tired of all these games and decide to have it out in a truel[1]. Out of habit, they take turns shooting (in alphabetical order).

1. Alice hits her target with probability $\frac{1}{2}$.
2. Bob hits his target with probability $\frac{3}{4}$.
3. Cameron never misses.

They continue this until only one of the three survives. Assuming all three use optimal strategies, what is the probability that Alice wins the truel?

[1] It's like a duel, but with 3 people.

Conventional Solution: Martin Gardner's Truel Problem

It's always a good idea to shoot at the person who's better at hitting you. So initially Alice and Bob try to shoot Cameron and Cameron tries to shoot Bob; eventually only two people are left and they shoot each other until one dies.

Notably, if Alice and Bob are left, then Alice has a probability of $\frac{4}{7}$ of winning when it's her turn to shoot, and $\frac{1}{7}$ chance of winning when it's Bob's turn to shoot.

We can construct a tree of outcomes and follow it to find probabilitities; after a bit of work, we get a probability of $\frac{39}{112}$, which is just over $\frac{1}{3}$.

Clever Solution: Martin Gardner's Truel Problem

Alice can improve her chances by shooting into the air when it's her turn and therefore not hitting anyone. This ensures that when someone dies, it will be her turn to shoot, guaranteeing her at least a $\frac{1}{2}$ chance of winning.

(Bob and Cameron can't get away with this because they get shot at when there are three people left; but nobody is going to shoot at Alice when there is a more threatening opponent around!)

Using a similar technique to find the probabilities, we see that Alice has a probability of $\frac{3}{7} + \frac{1}{8} = \frac{31}{56}$ of winning, which is better than $\frac{1}{2}$ even though Alice is the worst marksman of the three.