Project 1: e is irrational

Description

In this project, you will use a particular infinite series to prove that the number e is not a rational number.

Components

(a) Write a short introduction about the difference between rational and irrational numbers. List some examples of both types.

(b) Suppose that, instead, e actually *is* rational. That is, assume $\mathrm{e} = \frac{p}{q}$, for some positive integers p and q. Define the number M to be

$$M = q! \cdot \left(\mathrm{e} - \sum_{k=0}^{q} \frac{1}{k!} \right)$$

Explain why M must be a positive integer (i.e. not a fraction).

(Hint: Remember, we are assuming that $\mathrm{e} = \frac{p}{q}$ from here on out.)

(c) Now, explain why

$$M = q! \cdot \sum_{k=q+1}^{\infty} \frac{1}{k!} = \sum_{k=q+1}^{\infty} \frac{q!}{k!}$$

as well.

(d) Your goal now is to show that

$$M < \frac{1}{q}$$

To do this, you'll need to use the representation of M you just derived, and you'll want to compare it to a particular geometric series.

(This is the part that requires some ingenuity from you. See me if you need some hints!)

(e) Use the conclusion you just found to deduce that e is not a rational number.

(This is asking you to explain why the overall *technique* of the proof has succeeded. Again, see me if you need some hints.)

Extra Credit

(i) [1 point] Make your essay excellently written and understandable by your classmates.

(ii) [1 point] Look up a proof that $\sqrt{2}$ is irrational. (There's one on the Wikipedia page for $\sqrt{2}$, for instance.) Write that proof in your own words by following the steps. Compare the overall *structure* of that proof with the proof you completed here about e.

(iii) [1 point] Investigate and explain how you could perhaps modify the technique used here to also prove that e^2 is irrational. Think about why this is not immediately, obviously true, even after knowing that e is irrational.

Project 2: The Method of Undetermined Coefficients

Description

In this project, you will invent a new technique that can more efficiently solve particular integration problems. Specifically, when we want to integrate a trig function multiplied by an exponential function, we would usually have to do some complicated integration by parts. Here, you will discover a new and better way of doing these types of integrals.

Components

(a) Solve this integral using integration by parts:

$$I = \int \cos(3x) \cdot \mathrm{e}^{4x}\, dx$$

(Hint: You will need to do integration by parts twice. This will give you an equation where I is on both sides, written in terms of itself. Solve for I.)

(Feel free to use Wolfram or something to get the final expression and compare, to make sure you are correct. However, I will want to see your steps written out.)

(b) Now, consider a more general expression. Let A and B be arbitrary real numbers, and define J to be

$$J = \int \cos(Ax) \cdot \mathrm{e}^{Bx}\, dx$$

Someone who had *never seen* integration by parts might naively guess that the answer to this problem is a constant multiple of $\sin(Ax)\mathrm{e}^{Bx} + C$. Take a derivative to show them that their guess is wrong.

(c) Someone with a little more ingenuity might guess that the answer to the integral J is a *linear combination* of trig functions multiplied by an exponential function.

(In general, a *linear combination* of two expressions, X and Y, is something like $sX + tY$, where s and t are some constants.)

Looking at what you did in **(a)**, try to come up with an answer for the integral J. Take a derivative to make sure your answer is correct.

(d) Perform a similar analysis as above to come up with a method that can solve any integral of this form:

$$K = \int (Ax^2 + Bx + C) \cdot \mathrm{e}^{Dx}\, dx$$

where A, B, C, D are arbitrary constants.

That is, play around with an example to come up with a guess for what this answer is, in general. Then, take a derivative to make sure your guess is right.

Extra Credit

(i) [1 point] Make your essay excellently written and understandable by your classmates.

(ii) [2 points] Extend the technique you developed in **(d)** to analyze integrals of this form:

$$L = \int P(x) \cdot \mathrm{e}^{Ax}\, dx$$

where A is an arbitrary constant and $P(x)$ is *any polynomial* in x.

Project 3: Working with Power Series

Description

In this project, you will work with a particular power series. By manipulating it, you can discover what function it represents. Then, you will use it to generate a *new* infinite series for ln(2), one that we have not seen before!

Components

(a) Define $f(x)$ to be the power series

$$f(x) = \sum_{n=1}^{\infty} n \cdot x^n$$

Multiply out $(1 - x) \cdot f(x)$ directly and simplify the expression to discover the function $f(x)$ that the power series represents.

(b) What is the window of convergence for this power series? Could we use our power series to approximate $f(0.001)$? How about $f(0.5)$ How about $f(1.5)$? Which series approximations would converge more quickly? How do you know?

(c) Find the exact value of the integral

$$\int_0^{1/2} f(x)\, dx$$

using an integration technique we have studied in class.

(d) Use this exact value, and the infinite series representation of $f(x)$, to come up with an infinite series that equals $\ln(2)$.

(e) Do something similar to come up with an infinite series that equals $\ln(3)$.

Extra Credit

(i) [1 point] Make your essay excellently written and understandable by your classmates.

(ii) [1 point] Use a similar idea to what you did in **(a)** to come up with the function that is represented by this power series:

$$g_1(x) = \sum_{n=1}^{\infty} n(n+1) \cdot x^n$$

(iii) [1 point] Try to extend and generalize what you just did to come up with the functions that are represented by these kinds of power series:

$$g_2(x) = \sum_{n=1}^{\infty} n(n+1)(n+2) \cdot x^n$$

$$g_3(x) = \sum_{n=1}^{\infty} n(n+1)(n+2)(n+3) \cdot x^n$$

$$g_4(x) = \sum_{n=1}^{\infty} n(n+1)(n+2)(n+3)(n+4) \cdot x^n$$

and so on.

Project 4: Fourier Polynomials

Description

In this project, you will work with a particular kind of function that is a sum of trig functions. The French mathematician/physicist Joseph Fourier developed these functions in the 1800s when he was studying the properties of heat flow, because he realized that they are great for representing functions that are *periodic*. Here, you will use these functions to analyze some simple functions like x and x^2.

Components

First, some definitions. Suppose f is a continuous function defined on the interval $[-\pi, \pi]$. The **Fourier coefficients** of f are given by these formulae, defined for every $n = 0, 1, 2, 3, \ldots$:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx$$

These coefficients are used to define the **Fourier polynomials** that represent the function f. For every $N = 1, 2, 3, \ldots$, we define

$$P_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

(a) Take the function $f(x) = x$. Find all of the Fourier coefficients of f. That is, evaluate a_n and b_n for every $n = 0, 1, 2, 3, \ldots$.

(b) Find an expression for the first five Fourier polynomials of f. That is, write out what $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$, and $P_5(x)$ are. Then, plot all of these functions together, along with f, on one graph, over the interval $[-\pi, \pi]$.

Are these "good approximations" of f? Are they getting better? What do you think? What will happen with $P_6(x)$ and $P_7(x)$ and so on? (I'm not asking you to *prove* anything here; I just want you to share your opinion.)

(c) Something special happened with $f(x) = x$ because it is an *odd function*. Suppose $g(x)$ is *an arbitrary odd function* defined on $[-\pi, \pi]$. Make a guess about what the Fourier coefficients of g are. Prove that your guess is correct by using what we know about integrals and odd functions and the cosine function.

Extra Credit

(i) [1 point] Make your essay excellently written and understandable by your classmates.

(ii) [2 points] Repeat the above steps with the function $h(x) = x^2$. That is, find all of its Fourier coefficients; plot the first five Fourier polynomials on one graph with h and compare.

What is special about h? What property does it have? Can you make a guess about what happens with the Fourier coefficients of any function that also has this property? Can you prove it?

Project 5: The Famous "Fly Problem"

Description

There is a popular riddle/word problem about a fly traveling between two trains. There is a *very simple* solution to this problem that even a child could understand. There is also a more complicated solution that requires knowledge of infinite series. In this project, you will solve the puzzle on both ways.

A popular anecdote about the mathematician John von Neumann says that he was posed this particular puzzle at a party once, and he gave the correct answer immediately. The puzzle-poser asked him, "Oh, have you heard this one before?", and von Neumann replied, "No, I just summed the infinite series."

Components

Here is the puzzle:

> Two trains are traveling straight towards each other on the same track. They start at 60 kilometers apart, and they are both traveling at 10 kilometers per hour. A magical, speedy fly is traveling between them, back and forth, without ever stopping. He begins at the front of one train and flies at 20 kilometers per hour straight towards the other train. Once he reaches it, he immediately turns around and goes the other direction at the same speed. He repeats this forever and ever . . . until the two trains inevitably crash! How far did the fly travel, in total?

(a) Let D_n be the distance that the fly travels on his n-th trip between the two trains. Using the given information, figure out what D_1 and D_2 and D_3 and D_4 are.

(b) Make a guess at a formula for D_n.

(c) Use **mathematical induction** to prove that your guess for D_n is correct.

(You will need to see me for help with this, because we will not talk about mathematical induction in class. It is not difficult, we just won't have time to do it in class, so we'll need to talk about it in person.)

(d) Now that we know D_n, evaluate the infinite sum

$$\sum_{n=1}^{\infty} D_n$$

and conclude that this is how far the fly traveled.

(e) Go back and solve the problem in a really easy way, and see that you get the same answer. This method should be so simple that you can explain it to a friend, even someone who isn't studying mathematics. (Seriously, try it. Explain it to a friend or family member and see if they follow along.)

Extra Credit

(i) [1 point] Make your essay excellently written and understandable by your classmates.

(ii) [1 point] Include a one page description of the technique of *mathematical induction* so that a classmate reading your project can understand how this technique works.

(iii) [1 point] Generalize this result by considering a version of the puzzle where the two trains are hurtling towards each other at A kilometers per hour, and the fly travels between them at B kilometers per hour. What is the solution, and how does it depend on A and B? Are there any restrictions on what A and B are allowed to be for the puzzle to make sense?

Project 6: Solving a Differential Equation with a Power Series

Description

In this project, you will find a function that satisfies a certain *differential equation*. This just means that we have some knowledge about how a function's derivative depends on the function itself, and we can "unravel" this equation to find out what the function actually is. You will do this in two ways: (1) by using a power series representation of the function, and (2) by doing some integration. Then, you will compare and make sure that you get the same result, either way!

Components

We are looking to find a function f that has these two properties:

$$f(0) = 1 \qquad \text{and} \qquad f'(x) = 2x \cdot f(x) \quad \text{for every real number } x$$

(a) Take a few derivatives of the expression for $f'(x)$, and use these to evaluate $f'(0)$, $f''(0)$, $f'''(0)$, $f''''(0)$, and $f'''''(0)$. (That's the first five derivatives.)

(b) Make a guess for the value of the n-th derivative of f evaluated at 0. Use *mathematical induction* to prove that your guess is correct.

(You will need to see me for help with this, because we will not talk about mathematical induction in class. It is not difficult, we just won't have time to do it in class, so we'll need to talk about it in person.)

(c) Use what you now know about these values to write down the *Taylor series* for f, centered at $x = 0$. You now "know" the solution to the differential equation, in some sense.

(d) Solve the differential equation directly, by moving $f(x)$ to the other side and integrating.

(Hint: Don't forget the $+C$. Use the property $f(0) = 1$ somehow, too.)

(e) Use a Taylor series we derived in class to verify that the solutions you found in **(c)** and **(d)** are really the same function.

Extra Credit

(i) [1 point] Make your essay excellently written and understandable by your classmates.

(ii) [1 point] Include a one page description of the technique of *mathematical induction* so that a classmate reading your project can understand how this technique works.

(iii) [1 point] Generalize this result to solve this differential equation for the function g:

$$g(0) = 1 \qquad \text{and} \qquad g'(x) = x^k \cdot g(x)$$

where k is an arbitrary *positive integer*.

(You can approach this using the power series technique or the direct integration technique. I'll leave the choice up to you, but you *don't* need to do both and then compare, like you did above.)