Homework Assignment 4

Assigned Fri 2/28. Due Fri 3/20. 21-640 Spring 2020 (R. Pego)

1. Suppose that $\{u_n\}$ is a sequence in a Hilbert space H that converges weakly to $u \in H$: $u_n \rightharpoonup u$ as $n \to \infty$. If it is also true that $\|u_n\| \to \|u\|$, then prove $u_n \to u$ in the norm metric on H.

2. (RS III.16) A subset S of a Banach space X is called *weakly bounded* if each $\lambda \in X^*$ is bounded on S; that is, $\forall \lambda \in X^*$, $\sup_{x\in S} |\lambda(x)| < \infty$. S is called *strongly bounded* if $\sup_{x\in S} \|x\| < \infty$. Prove that a set is strongly bounded if and only if it is weakly bounded.

3. Suppose that S is a bounded subset of ℓ_2 and let $\overline{S}^w$ denote the weak closure of S (the smallest subset of ℓ_2 that contains S and is closed in the weak topology on ℓ_2). Show that if $x \in \overline{S}^w$ then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in S that converges weakly to x: $x_n \rightharpoonup x$ as $n \to \infty$.

4. Suppose $\{x_n\}$ is sequence in a Banach space X converging weakly to $x \in X$ and $\{\ell_n\}$ is a sequence in the dual X^* converging weak-$*$ to $\ell \in X^*$.
   (a) Give an example to show that it is not necessarily true that $\ell_n(x_n) \to \ell(x)$ as $n \to \infty$.
   (b) Prove that if either (x_n) or (ℓ_n) converges strongly, then necessarily $\ell_n(x_n) \to \ell(x)$ as $n \to \infty$.

5. (RS IV.40) Let X be an infinite-dimensional Banach space with the weak topology. Prove that the (weak) closure of the unit sphere $S = \{x \in X : \|x\| = 1\}$ is the unit ball $B = \{x \in X : \|x\| \le 1\}$.

6. In H^1_{per}, the Hilbert-space completion of the space $\mathcal{S}$ of 2π-periodic trig polynomials on $\mathbb{R}$ with inner product
$$(f,g)_{H^1} = \int_{-\pi}^{\pi} \overline{f(x)}g(x) + \overline{f'(x)}g'(x)\,dx,$$
suppose $\{f_k\}_{k\in\mathbb{N}}$ is a sequence which converges *weakly* to some $f \in H^1_{\text{per}}$. I.e.,
$$(f_k, g)_{H^1} \to (f,g)_{H^1} \qquad \text{for all } g \in H^1_{\text{per}}.$$
Prove that $f_k \to f$ *uniformly*.

7. [Optional: do not turn in.] Suppose that $\{x_\alpha\}_{\alpha\in I}$ is a *Hamel basis* for a Banach space X; i.e., X is the algebraic span of the set $\{x_\alpha : \alpha \in I\}$. Thus every $x \in X$ has a unique representation as
$$x = \sum_{\alpha\in I} c_\alpha x_\alpha\,, \quad c_\alpha \in \mathbb{C}\,,$$
where only finitely many terms are nonzero. It is straightforward to show that for each $\alpha \in I$ the "coordinate map" $x \mapsto c_\alpha$ is linear. Prove that only finitely many of these maps can be continuous. [Hint: if $\{\alpha_j : j \in \mathbb{N}\}$ is a sequence in I, consider the sequence of maps $x \mapsto T_n x = \sum_{j\le n} j c_{\alpha_j} x_{\alpha_j}$, and carefully interpret the statement of the uniform boundedness principle.]