Homework Assignment 3

Assigned Fri 2/14. Due Wed 2/26. 21-640 Spring 2020 (R. Pego)

Below we use the notation $\boldsymbol{a} = \{a_k\}_{k\in\mathbb{N}}$ to represent a sequence of complex numbers, and set

$$c_0 = \{\boldsymbol{a} \mid \lim_{k\to\infty} a_k = 0\}, \qquad \ell_\infty = \{\boldsymbol{a} \mid \|\boldsymbol{a}\|_\infty := \sup_k |a_k| < \infty\},$$

$$\ell_p = \{\boldsymbol{a} \mid \|\boldsymbol{a}\|_p := \left(\sum_{k=1}^\infty |a_k|^p\right)^{1/p} < \infty\} \quad (p > 0).$$

1. (RS III.2a) If $p \geq 1$, prove ℓ_p and c_0 are separable, but ℓ_∞ is not.

2. (a) Prove that $\ell_1^* = \ell_\infty$.

   (b) Using the Hahn-Banach theorem, prove there is a linear functional T on ℓ_∞ with the property that for all $\boldsymbol{a} \in \ell_\infty$ whose components a_j are all real,

   $$\liminf_{n\to\infty} a_n \leq T(\boldsymbol{a}) \leq \limsup_{n\to\infty} a_n.$$

   [Hint: Taking limits is a linear operation. Or is it?]

   (c) Deduce that the dual $\ell_\infty^* \neq \ell_1$. I.e., deduce that not every bounded linear functional $S : \ell_\infty \to \mathbb{C}$ has the form $S\boldsymbol{a} = \sum b_j a_j$ for some sequence $\boldsymbol{b} = \{b_j\} \in \ell_1$.

3. Let X be a Banach space and let (ε_n) be a sequence of positive numbers converging to zero. Suppose that (f_n) is a sequence in the dual X^* having the property that for all $x \in X$ with $\|x\| < 1$, there exists $C(x) > 0$ such that

   $$|f_n(x)| \leq \varepsilon_n \|f_n\| + C(x).$$

   Prove that (f_n) is bounded. [Hint: let $g_n = f_n/(1 + \varepsilon_n\|f_n\|]$

4. Let X be a Banach space and suppose $T : X \to X^*$ is linear and has the property that whenever $f = T(x)$ then $f(x) \geq 0$. Prove T is bounded.

5. (a) For any $u \in L^2_{\text{per}}$, the Hilbert-space completion of the space of 2π-periodic trigonometric polynomials with inner product $(u, v) = \int_{-\pi}^{\pi} \overline{u(x)} v(x)\, dx$, define $\hat{u}(k) = (e_k, u)$, $e_k(x) = e^{ikx}/\sqrt{2\pi}$. Prove Plancherel's identity:

   $$(u, v) = \sum_{k\in\mathbb{Z}} \overline{\hat{u}(k)} \hat{v}(k).$$

   (b) (The isoperimetric inequality) Suppose we have a smooth closed curve in the (complex) plane which encloses an area A and has perimeter P. We wish to prove that

   $$P^2 \geq 4\pi A\ . \tag{$**$}$$

   To do this, assume that the curve is parametrized by a smooth 2π-periodic complex valued function $f(x) = u(x) + iv(x)$ such that $(u')^2 + (v')^2 = c^2$ is constant. Using that $c((u')^2 + (v')^2)^{1/2} = |f'|^2$, relate P^2 to $\int_{-\pi}^{\pi} |f'(x)|^2\, dx$. Relate $A = \int u\, dv$ to the L^2-inner product

   $$(f', f) = \int_{-\pi}^{\pi} \overline{f'(x)} f(x)\, dx.$$

   Using Plancherel's identity you should be able to deduce $(**)$.

6. (a) Show that the 2π-periodic function $u(x) = |\sin \frac{x}{2}|$ belongs to the periodic Sobolev space $H = H^1_{\text{per}}$, and is a weak solution of a problem of the form

$$-u''(x) + u(x) = f,$$

where $f \in H^*$ is a linear combination of the Dirac delta distribution and a map of the form $v \mapsto \int_0^{2\pi} \overline{g(x)} v(x)\, dx$.

(b) Show that the function $w(x) = \sqrt{u(x)}$ does not belong to H^1_{per}. In particular, you must show that w has *no* weak derivative in L^2_{per}. That is, *there is no linear map* $T : C^\infty_{\text{per}} \to \mathbb{C}$ that is bounded with respect to the L^2 norm such that

$$T(f) = -\int_0^{2\pi} \overline{w(x)} f'(x)\, dx \quad \text{for all } f \in C^\infty_{\text{per}}.$$

[Q: Why must such a map necessarily take the form you naturally guess?]