Homework Assignment 2

Assigned Mon 2/3. Due Wed 2/12. 21-640 Spring 2020 (R. Pego)

1. (cf. Reed-Simon p63 #4b) Suppose V is a normed vector space over $\mathbb{C}$ whose norm satisfies the *parallelogram law*

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \text{for all } u, v \in V.$$

Show then that the polarization identity

$$(u,v) := \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 - i\|u+iv\|^2 + i\|u-iv\|^2\right)$$

defines an inner product on V.

2. (Reed-Simon p63 #5) Let V be an inner product space over $\mathbb{C}$ and let $\{x_1, \ldots, x_N\}$ be an orthonormal set. Given any $x \in H$, prove that

$$\left\| x - \sum_{1 \le j \le N} a_j x_j \right\|$$

is minimized by choosing $a_j = (x_j, x)$ for all $j \in \{1, \ldots, N\}$.

3. Reed-Simon p33 #11ab. (See page 10 for the definition of the normed vector space $S[a,b]$ consisting of step functions on $[a,b]$ taken with the supremum norm.)

4. Let H be a separable Hilbert space and let $A = \{x_1, x_2, \ldots\}$ be a countable orthonormal basis. (Recall this means $\overline{\operatorname{span} A} = H$.) Suppose $B = \{y_1, y_2, \ldots\}$ is an orthonormal set such that

$$\sum_{j=1}^{\infty} \|x_j - y_j\|^2 < \infty.$$

   (a) Suppose $z \in H$ and $z \in B^\perp$, i.e., $(z, y_j) = 0$ for all j. Show that if $\sum_{j>N} \|x_j - y_j\|^2 < 1$ then $z \in \operatorname{span}\{y_1, \ldots, y_N\}$.

   (b) Show that B is an orthonormal basis of H.

5. Let V be a subspace of a Hilbert space H, and suppose $T \in \mathcal{L}(V, \mathbb{C})$ is a bounded linear functional on V, with least bound M on V defined in the usual sense:

$$M := \sup_{\substack{u \in V \\ u \neq 0}} \frac{|Tu|}{\|u\|} < \infty.$$

Prove that there exists a *unique* extension of T in $\mathcal{L}(H, \mathbb{C})$ that has the *same* least bound on H, so $\|T\| = M$. (Note V may be neither closed nor dense.)