# Homework Assignment 1

Assigned Wed 1/22. Due Fri 1/31. 21-640 Spring 2020 (R. Pego)

1. Let $\ell^2$ denote the vector space of complex sequences $\boldsymbol{x} = \{x_k\}_{k=1}^\infty$ such that $\|\boldsymbol{x}\| < \infty$, where $\|\boldsymbol{x}\|^2 = \sum_{k=1}^\infty |x_k|^2$. Prove that $\ell^2$ is complete. (Hence it is a Hilbert space.)

2. Any compact set $S$ in a metric space must be closed and bounded. Moreover, for any $\boldsymbol{x} = \{x_k\}_{k=1}^\infty \in \ell^2$ we have
$$\sum_{k=N}^\infty |x_k|^2 \to 0 \quad \text{as } N \to \infty. \tag{1}$$
Suppose $S$ is a closed and bounded subset of $\ell^2$. Prove: $S$ is compact if and only if (1) holds *uniformly* for all $\boldsymbol{x} \in S$.

3. (Reed-Simon p34 #27) Use an $\varepsilon/3$ argument to prove the following: Let $X$ be a complete normed linear space. Suppose that for each $n \in \mathbb{N}$, $T_n : X \to X$ is linear, and these maps are uniformly bounded, i.e., $M := \sup_n \|T_n\| < \infty$. Suppose further that the sequence $(T_n x)$ converges for each $x$ in a set $D$ that is *dense* in $X$. Then the maps $T_n$ have a pointwise limit $T$ for every $x \in X$. (It was shown in class that such a limit determines a bounded linear map.)

4. (20 points) Let $a > 0$ and let $E$ be the Banach space $C[0, a]$ of continuous functions $u : [0, a] \to \mathbb{C}$, with norm $\|u\|_\infty = \sup_{x\in[0,a]} |u(x)|$. Define $T : E \to E$ by
$$(Tu)(x) = \int_0^x u(s)\, ds.$$
   (a) Find $\|T\|$.
   (b) Show that $T$ is injective. Also show $T$ is not surjective, and that $T^{-1}$ is *not* bounded from the image range$(T) \to E$. (Remark: If $E = \mathbb{C}^N$, any injective linear $T : E \to E$ is automatically bounded and surjective with bounded inverse.)
   (c) Find $\|T^n\|$ for $n = 2, 3, \ldots$, and show that $T$ is *quasinilpotent*, meaning
$$\|T^n\|^{1/n} \to 0 \quad \text{as } n \to \infty.$$
   (d) Show that $T$ has *no eigenvalues*. I.e., show that for any $\lambda \in \mathbb{C}$ and $u \in E$, if $Tu = \lambda u$ then $u = 0$.