L'Hôpital's rule practice problems

21-121: Integration and Differential Equations

Find the following limits. You may use L'Hôpital's rule where appropriate. Be aware that L'Hôpital's rule may not apply to every limit, and it may not be helpful even when it does apply. Some limits may be found by other methods. These problems are given in no particular order. (Where appropriate, sources for the problems are given in square brackets under the answer. See the end for an explanation of these references.)

1. $\lim_{x\to 0} \dfrac{e^x - x - 1}{\cos x - 1}$ *Ans.* -1. [R 4.7 Ex. 4]

2. $\lim_{x\to -2} \dfrac{x^3 - x^2 - 10x - 8}{5x^3 + 12x^2 - 2x - 12}$ *Ans.* $\frac{3}{5}$.

3. $\lim_{x\to a} \dfrac{x - a}{\ln x - \ln a}$ *Ans.* a. [SSJ 124.29]

4. $\lim_{\theta\to 0} \dfrac{1 - \cos\theta}{\theta^2}$ *Ans.* $\frac{1}{2}$. [M 62.12]

5. $\lim_{x\to\infty} (\sinh x - \cosh x)$ *Ans.* 0.

6. $\lim_{x\to \pi/2^+} \dfrac{\tan x}{\ln(2x - \pi)}$ *Ans.* ∞. [W VIII.2.1]

7. $\lim_{x\to\infty} \dfrac{\tanh x}{\tan^{-1} x}$ *Ans.* $\frac{2}{\pi}$.

8. $\lim_{x\to 0} \dfrac{\sin x}{\sinh x}$ *Ans.* 1.

9. $\lim_{x\to 0} \left(\dfrac{e^x}{e^x - 1} - \dfrac{1}{x}\right)$ *Ans.* $\frac{1}{2}$. [R Ch. 4 rev. 116]

10. $\lim_{x\to \pi/2} \dfrac{\ln(x - \pi/2)}{\sec x}$ *Ans.* 0. [SSJ 122 Ex. 4]

11. $\lim_{x\to 2} \dfrac{e^{x^2} - e^4}{x - 2}$ *Ans.* $4e^4$. [R 4.7.22]

12. $\lim_{x\to\infty} \dfrac{\sqrt[3]{x}}{\ln x}$ *Ans.* ∞.

13. $\lim_{x\to \pi/4} \dfrac{\ln(\tan x)}{\sin x - \cos x}$ *Ans.* $\sqrt{2}$. [Sh 9.1.6(b)]

14. $\lim_{x\to\infty} \dfrac{e^{-x}}{\sin x}$ *Ans.* Does not exist. [H 4.8.18(c)]

15. $\lim_{x\to\infty} \dfrac{e^{-x}}{\sin x + 2}$ *Ans.* 0.

16. $\lim_{x\to 1} \dfrac{\tan^{-1} x - \frac{\pi}{4}}{\tan \frac{\pi}{4}x - 1}$ *Ans.* $\frac{1}{\pi}$. [R 4.7.51]

17. $\lim_{x\to -\infty} \dfrac{\tan^{-1} x}{\cot^{-1} x}$ *Ans.* $-\frac{1}{2}$. [W VIII.1.7]

18. $\lim_{x\to 0} \sin x \ln x$ *Ans.* 0. [M 62.32]

19. $\lim_{x\to 0} x \sin \dfrac{1}{x}$ *Ans.* 0.

20. $\lim_{x\to \pi/2} \dfrac{\tan 3x}{\tan 5x}$ *Ans.* $\frac{5}{3}$. [R 4.7.18]

21. $\lim_{x\to \pi/2^+} \dfrac{\sec x}{\ln \sec x}$ *Ans.* $-\infty$. [M 62.26]

22. $\lim_{x\to 0} \left(\csc x - \dfrac{1}{x}\right)$ *Ans.* 0.

23. $\lim_{x\to \pi/2} (\tan x)^{\sin 2x}$ *Ans.* 1. [S 21.3(b)]

24. $\lim_{x\to -\infty} x^{-3} e^x$ *Ans.* 0.

25. $\lim_{x\to\infty} \dfrac{\ln(t + 2)}{\log_2 t}$ *Ans.* $\ln 2$. [R Ch. 7 Rev. 115]

26. $\lim_{x\to e} \dfrac{1 - \ln x}{x/e - 1}$ *Ans.* -1.

27. $\lim_{x\to\pi} \dfrac{\sqrt{1 - \tan x} - \sqrt{1 + \tan x}}{\sin 2x}$ *Ans.* $-\frac{1}{2}$. [Sh 9.1.14]

28. $\lim_{x\to\infty} \dfrac{(\pi/2) - \tan^{-1} x}{x^{-1}}$ *Ans.* 1. [W VIII.1.3 Ex. H]

29. $\lim_{x\to -1} \dfrac{\sqrt{x + 10} + 3x^{1/3}}{4x^2 + 3x - 1}$ *Ans.* $-\frac{7}{30}$.

30. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$ *Ans.* $-\frac{1}{6}$. [SM 7.6.13]

31. $\lim_{x\to 0} \dfrac{\tan x - x}{x^3}$ *Ans.* $\frac{1}{3}$. [SM 7.6.14]

32. $\lim_{x\to 2} \dfrac{x^4 - 4^x}{\sin(\pi x)}$ *Ans.* $\dfrac{32(1 - \ln 2)}{\pi}$.

33. $\lim_{x\to 3\pi} \dfrac{1 + \tan(x/4)}{\cos(x/2)}$ *Ans.* 1. [W VIII.1.2]

34. $\lim_{x\to\infty} x^{1/(\ln x)}$ *Ans.* e.

35. $\lim_{x\to 0} (\cos x)^{\csc x}$ *Ans.* 1.

36. $\lim_{x\to \pi/2^-} \cos x \ln \tan x$ *Ans.* 0. [SSJ 124.17]

37. $\lim_{x\to\infty} x^{\sin(1/x)}$ *Ans.* 1.

38. $\displaystyle\lim_{x\to\pi/2}\left[\left(\frac{\pi}{2}-x\right)\tan x\right]$ *Ans.* 1.
[SSJ 124.24]

39. $\displaystyle\lim_{x\to 0+}\frac{\sin 3x\cot 2x}{\ln\cos x}$ *Ans.* $-\infty$.

40. $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cot x}{x}\right)$ *Ans.* $\dfrac{1}{3}$.
[S 21.1(e)]

41. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x^2\csc x}$ *Ans.* 1.

42. $\displaystyle\lim_{x\to 1}\tan\left(\frac{\pi x}{2}\right)\ln x$ *Ans.* $-\dfrac{2}{\pi}$.
[R 4.7.26]

43. $\displaystyle\lim_{x\to -2}\frac{xe^x-4+2e^x-2x}{1+x\sin\pi x+x/2+2\sin\pi x}$
Ans. $2e^{-2}-4$.

44. $\displaystyle\lim_{x\to 1^-}\sqrt{1-x}\,\ln\ln(1/x)$ *Ans.* 0.
[W VIII.3.20]

45. $\displaystyle\lim_{x\to 0^+}(\csc x)^{(\cos^{-1}x)/(\ln x)}$ *Ans.* $e^{-\pi/2}$.

46. $\displaystyle\lim_{\theta\to 0}\frac{2\sin\theta-\sin 2\theta}{\sin\theta-\theta\cos\theta}$ *Ans.* 3.
[R Ch. 4 rev. 113]

47. $\displaystyle\lim_{x\to\infty}\frac{e^x+x}{\sinh x}$ *Ans.* 2.

48. $\displaystyle\lim_{\theta\to\pi/2}\frac{2(e^{\cos\theta}+\theta-1)-\pi}{\ln\sin(-3\theta)}$ *Ans.* $-\dfrac{2}{9}$.
[SSJ 124.9]

49. $\displaystyle\lim_{x\to 0^+}\frac{x\sin(1/x)}{\ln x}$ *Ans.* 0.

50. $\displaystyle\lim_{x\to 0^+}\frac{x\cot x}{e^x-1}$ *Ans.* ∞.

51. $\displaystyle\lim_{x\to 0^+}\frac{x\ln x}{\ln(1+ax)},\quad a>0$ *Ans.* $-\infty$.
[W VIII.1.10]

52. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x\cos^{-1}x}$ *Ans.* $\dfrac{2}{\pi}$.

53. $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{\cot x}$ *Ans.* 0.
[SM 7.6.31]

54. $\displaystyle\lim_{x\to\infty}(ax)^{b/(cx)},\quad a,c\neq 0$ *Ans.* 1.

55. $\displaystyle\lim_{x\to\infty}\frac{\tan^{-1}x-\pi/2+x^{-1}}{\coth^{-1}x-x^{-1}}$ *Ans.* 1.
[W VIII.4.4]

56. $\displaystyle\lim_{x\to+\infty}\frac{x^{10^{10}}}{e^x}$ *Ans.* 0.
[W VIII.1.5]

$\star$ **57.** $\displaystyle\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{1/x^2}$ *Ans.* $e^{-1/6}$.
[R 4.7.75(a)]

$\star$ **58.** $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)$ *Ans.* $\dfrac{1}{3}$.
[R 4.7.75(b)]

$\star\star$ **59.** $\displaystyle\lim_{x\to+\infty}\frac{x^{\sqrt{\ln x}}\left(\sqrt{\ln x}\right)^x}{\left(\sqrt{x}\right)^{\ln x}(\ln x)^{\sqrt{x}}}$ *Ans.* ∞.
[W VIII.4.11]

60. Compute $\displaystyle\lim_{x\to 0}\frac{x^2\cos x}{2\sin^2\frac{1}{2}x}$ by two methods.
Ans. 2.
[M 62.10]

61. Finding the following limit was the first example that L'Hôpital gave in demonstrating the rule that bears his name. Let a be a positive constant and find

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}.$$

Ans. 0.
[Sh 9.1.16]

62. Show that L'Hôpital's rule applies to the limit $\displaystyle\lim_{x\to\infty}\frac{x+\cos x}{x-\cos x}$, but that it is of no help. Then evaluate the limit directly.
Ans. 1.
[R 4.7.69]

$\star$ **63.** Without using L'Hôpital's rule, the limit $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$ can be evaluated by a rather intricate geometric argument. Show that it can be evaluated easily using L'Hôpital's rule. Then explain why doing so involves *circular reasoning.* [R 4.7.72]

References:

[H] Deborah Hughes-Hallett, Andrew M. Gleason, William G. McCallum, et al. *Calculus: Single and Multivariable*, second edition. Wiley, 1998.

[M] Ross R. Middlemiss. *Differential and Integral Calculus*, first edition. McGraw-Hill, 1940.

[R] Jon Rogawski. *Calculus: Early Transcendentals.* Freeman, 2008.

[S] Ivan S. Sokolnikoff. *Advanced Calculus.* McGraw-Hill, 1939.

[Sh] Shahriar Shahriari. *Approximately Calculus.* American Mathematical Society, 2006.

[SM] Robert T. Smith and Roland B. Minton. *Calculus*, second edition. McGraw-Hill, 2002.

[SSJ] Edward S. Smith, Meyer Salkover, and Howard K. Justice. *Calculus.* Wiley, 1938.

[W] David V. Widder. *Advanced Calculus.* Prentice-Hall, 1947.