3.5.9. The Mythic Forge & Steel Co. (MF&S) supplies castings to a variety of customers. It plans to devote the next week of production to just two, James Manufacturing and Woolcott Enterprises. MF&S uses a combination of pure steel and scrap metal to fulfill its orders and has 400 pounds of pure steel and 360 pounds of scrap metal in stock. The pure steel costs MF&S $6 per pound and the scrap $3 per pound. Pure steel requires 3 hours per pound to process into a casting, while scrap requires only 2 hours per pound. Total available processing time in the week is 2,000 hours.

The castings for James each require 5 pounds of metal, with a quality control restriction limiting the ratio of scrap to pure steel to a maximum of $\frac{5}{7}$. James has ordered 30 castings at a price of $50 each.

The castings for Woolcott each require 8 pounds of metal, with a quality restriction of a maximum scrap-to-pure-steel ratio of $\frac{2}{3}$. Woolcott has ordered 40 castings at a price of $80 each.

Determine how MF&S should allocate their metal stocks to produce the castings ordered by these two customers if the objective is to maximize the value added to the metal, i.e., to maximize the selling price minus the cost of the metal.

Variables. The first thing to consider is what our variables are. Think about the decisions in the problem, and what an answer to the problem would look like. In this case, an answer to the problem would look like the following:

> MF&S should allocate ______ pounds of pure steel and ______ pounds of scrap metal to the castings for James, and ______ pounds of pure steel and ______ pounds of scrap metal to the castings for Woolcott.

This is the form of a full and complete answer to the question posed in the problem. These blanks should contain numbers. Those numbers are the decisions we must make in order to answer the question. Therefore, we should define a variable for each of these blanks.

So we'll define the following variables, being very careful to define precisely what each of the variables means:

- Let PJ be the number of pounds of pure steel allocated to the castings for James.
- Let SJ be the number of pounds of scrap metal allocated to the castings for James.
- Let PW be the number of pounds of pure steel allocated to the castings for Woolcott.
- Let SW be the number of pounds of scrap metal allocated to the castings for Woolcott.

Notice that we are using descriptive names for the variables, rather than nondescriptive names like x_1, x_2, x_3, and x_4.

Variable domains. It is important to specify the domains of the variables, that is, whether each variable is nonnegative, nonpositive, or unrestricted. This is easy to forget, so let's do it now. In this case, all of our variables must be nonnegative, so we have the variable domains $PJ \geq 0$, $SJ \geq 0$, $PW \geq 0$, and $SW \geq 0$.

Objective function. Now that we have defined our variables, we should consider the objective function. The objective function is the quantity that we are aiming to minimize or maximize. In this problem, our objective is to maximize the selling price minus the cost of the metal.

We know that the total selling price of the castings for James will be $30 \times \$50 = \$1500$, and the total selling price of the castings for Woolcott will be $40 \times \$80 = \$3200$, so the total selling price of all of the castings will be $\$1500 + \$3200 = \$4700$.

Each pound of pure steel that MF&S uses costs $6. There will be PJ pounds of pure steel used for James and PW pounds of pure steel used for Woolcott, so the total number of pounds of pure steel used will be $PJ + PW$. Therefore, the total cost of the pure steel used will be $6(PJ + PW) = 6PJ + 6PW$ dollars.

Similarly: Each pound of scrap metal that MF&S uses costs $3. There will be SJ pounds of scrap metal used for James and SW pounds of scrap metal used for Woolcott, so the total number of pounds of scrap metal used will be $SJ + SW$. Therefore, the total cost of the scrap metal used will be $3(SJ + SW) = 3SJ + 3SW$ dollars.

This means that the total cost of all of the metal used will be $6PJ + 6PW + 3SJ + 3SW$ dollars.

Our objective is to maximize the selling price minus the cost of the metal, so we want to maximize $4700 - 6PJ - 6PW - 3SJ - 3SW$. That is our objective function.

Constraints. To identify the constraints, let's go through the problem sentence by sentence.

> The Mythic Forge & Steel Co. (MF&S) supplies castings to a variety of customers. It plans to devote the next week of production to just two, James Manufacturing and Woolcott Enterprises. MF&S uses a combination of pure steel and scrap metal to fulfill its orders ...

Okay, so far this is just introductory information about the company.

> ... and has 400 pounds of pure steel and 360 pounds of scrap metal in stock.

This is a description of two resource constraints. The total amount of pure steel MF&S uses, which is given by $PJ + PW$, must not exceed 400 pounds; therefore, we have the constraint $PJ + PW \leq 400$. Likewise, the total amount of scrap metal MF&S uses, which is given by $SJ + SW$, must not exceed 360 pounds; therefore, we have the constraint $SJ + SW \leq 360$.

> The pure steel costs MF&S \$6 per pound and the scrap \$3 per pound.

We used this information to formulate the objective function.

> Pure steel requires 3 hours per pound to process into a casting, whil scrap requires only 2 hours per pound. Total available processing time in the week is 2,000 hours.

This describes another resource constraint; the resource here is processing time. As with all resources, the amount of this resource that MF&S uses must not exceed the amount that is available, so the form of the resource constraint should be (amount used) $\leq$ (amount available). The amount of processing time available is clearly stated: 2,000 hours. To find the amount of processing time used, break it up into the amount used for the pure steel and the amount used for the scrap metal. The total amount of pure steel used is $PJ + PW$ pounds, and each pound of pure steel requires 3 hours of processing time, so the amount of processing time used for the pure steel is $3(PJ + PW) = 3PJ + 3PW$ hours. Likewise, the total amount of scrap metal used is $SJ + SW$ pounds, and each pound of scrap metal requires 2 hours of processing time, so the amount of processing time used for the scrap metal is $2(SJ + SW) = 2SJ + 2SW$ hours. Therefore, the total amount of processing time required for all of the metal is $3PJ + 3PW + 2SJ + 2SW$ hours, and this must not exceed the amount of this resource that we have available, which is 2,000 hours. This means that we have the constraint $3PJ + 3PW + 2SJ + 2SW \leq 2{,}000$.

> The castings for James each require 5 pounds of metal .... James has ordered 30 castings at a price of \$50 each.

We used the price information to formulate the objective function. But these sentences also give us information about the total weight of metal that must be used for the castings for James: $30 \times 5 = 150$ pounds. Now, the amount of metal that is to be allocated to these castings is $PJ + SJ$ pounds, so we must have the equality constraint $PJ + SJ = 150$.

> The castings for James ... [have] a quality control restriction limiting the ratio of scrap to pure steel to a maximum of $\frac{5}{7}$.

The amount of scrap metal used for the castings for James is SJ pounds, and the amount of pure steel used for the castings for James is PJ pounds, so the ratio of scrap metal to pure steel for these castings is SJ/PJ. The requirement is that this ratio be no greater than $\frac{5}{7}$, so we have the constraint $SJ/PJ \leq \frac{5}{7}$. As written, this is not a linear inequality, because we are dividing one variable by another. But we can multiply both sides of the inequality by PJ (which is nonnegative, so the inequality sign does not flip) to get the linear inequality $SJ \leq \frac{5}{7}PJ$.

> The castings for Woolcott each require 8 pounds of metal, with a quality restriction of a maximum scrap-to-pure-steel ratio of $\frac{2}{3}$. Woolcott has ordered 40 castings at a price of $80 each.

Exactly like the previous paragraph: The total amount of metal that must be used for the Woolcott castings is $40 \times 8 = 320$ pounds, and the amount of metal that is to be allocated to these castings is $PW + SW$ pounds, so we must have the equality constraint $PW + SW = 320$. These castings are to be made from SW pounds of scrap metal and PW pounds of pure steel, so the ratio of scrap metal to pure steel is SW/PW. This ratio must be no larger than $\frac{2}{3}$, so we have the constraint $SW/PW \leq \frac{2}{3}$. In order to write this as a linear inequality, we multiply both sides by the (nonnegative) quantity PW to get $SW \leq \frac{2}{3}PW$.

> Determine how MF&S should allocate their metal stocks to produce the castings ordered by these two customers if the objective is to maximize the value added to the metal, i.e., to maximize the seling price minus the cost of the metal.

This paragraph is telling us what our objective function is (and, before that, what questions we are asked to answer, so that we know what our variables should be).

That's every sentence in the problem, so that takes care of the constraints. Putting everything together now, we get the following linear program:

$$
\begin{array}{lll}
\text{maximize} & 4700 - 6PJ - 6PW - 3SJ - 3SW & \\
\text{subject to} & PJ + PW \leq 400 & \text{[pure steel in stock]} \\
& SJ + SW \leq 360 & \text{[scrap metal in stock]} \\
& 3PJ + 3PW + 2SJ + 2SW \leq 2{,}000 & \text{[processing time]} \\
& PJ + SJ = 150 & \text{[enough metal for James]} \\
& SJ \leq \dfrac{5}{7}PJ & \text{[quality restriction for James]} \\
& PW + SW = 320 & \text{[enough metal for Woolcott]} \\
& SW \leq \dfrac{2}{3}PW & \text{[quality restriction for Woolcott]} \\
& PJ \geq 0, \quad SJ \geq 0, \quad PW \geq 0, \quad SW \geq 0. &
\end{array}
$$

Implementation notes. What we have above is a perfectly fine linear programming formulation of the problem. If we want to actually solve this linear program, however, we will have to make two small changes.

First, our objective function contains a constant term, 4700. The simplex algorithm is not designed to work with objective functions containing constant terms. But because 4700 is a constant, this is not a problem: we will just subtract the 4700 term from the objective function, maximize the quantity $-6PJ - 6PW - 3SJ - 3SW$ instead (which is 4700 less than the quantity we really want to maximize), and remember to add 4700 back to the optimal objective value at the end.

(The textbook's formulation for this problem takes a different approach here. Rather than subtracting the total cost of the metal from the total selling price, as we did, the textbook works with each pound of metal individually. Each pound of pure steel allocated to the castings for James costs $6 and sells for $10, because the 5-pound castings for James sell for $50; so the value added for each pound of pure steel allocated for James is $\$10 - \$6 = \$4$, and therefore the total value added for all of the pure steel allocated for James is $4PJ$ dollars. Similarly, the total value added for the scrap metal allocated for James is $7PW$ dollars, the total value added for the pure steel allocated for Woolcott is $4PW$ dollars, and the total value added for the scrap metal allocated for Woolcott is $7SW$ dollars. Therefore, the total value added altogether is $4PJ + 7PW + 4PW + 7SW$ dollars, and this is what the textbook uses as the objective function. This will be equal to our objective function, $4700 - 6PJ - 6PW - 3SJ - 3SW$, for any feasible solution.)

Second, some of our constraints have variables on the right-hand side of the inequality. We will need to move these to the left-hand side before we apply the simplex algorithm. So the constraint $SJ \leq \frac{5}{7}PJ$ becomes $SJ - \frac{5}{7}PJ \leq 0$, and the constraint $SW \leq \frac{2}{3}PW$ becomes $SW - \frac{2}{3}PW \leq 0$. (If we like, we can multiply these constraints by a constant to clear the fractions; so $SJ - \frac{5}{7}PJ \leq 0$ becomes $7SJ - 5PJ \leq 0$ after we multiply by 7, and $SW - \frac{2}{3}PW \leq 0$ becomes $3SW - 2PW \leq 0$ after we multiply by 3. But doing this will not necessarily make the simplex algorithm easier, because once we start pivoting we will probably have to divide by these constants and come back to fractions anyway.)

After we make these small changes and rearrange the constraints to line up the variables in columns, we get the linear program

$$\begin{array}{rrrrrl}
\text{maximize} & -6PJ - 3SJ - 6PW - 3SW & & & & \\
\text{subject to} & PJ & & + PW & & \leq 400 \\
& & SJ & & + SW & \leq 360 \\
& 3PJ & + 2SJ & + 3PW & + 2SW & \leq 2{,}000 \\
& PJ & + SJ & & & = 150 \\
& -\frac{5}{7}PJ & + SJ & & & \leq 0 \\
& & & PW & + SW & = 320 \\
& & & -\frac{2}{3}PW & + SW & \leq 0 \\
\end{array}$$

$$PJ \geq 0, \quad SJ \geq 0, \quad PW \geq 0, \quad SW \geq 0.$$